AMERICAN JOURNAL of PHYSICS

A Journal Devoted to the Instructional and Cultural Aspects of Physical Science

VOLUME 19, NUMBER 2 FEBRUARY, 1951

On an Application of Dimensional Analysis

S. F. BORG
U. S. Naval Postgraduate School, Annapolis, Maryland
(Received June 2, 1950)

Based upon dimensional reasoning and utilizing the known invariants of physical tensors, it is shown that the complete forms (except for multiplying constants) of certain differential and integral equations of applied mechanics can be obtained. The procedure equates physical and tensor invariants in a basic postulated form of equation. Just as in ordinary dimensional analysis, the physical terms which enter into any given problem must be known beforehand. In this case, these terms are tensors in addition to the usual physical invariants such as pressure, volume, energy, modulus of elasticity, etc.

A FORM of dimensional reasoning is described and various applications to the fields of applied mechanics are indicated. Fundamentally, the method depends upon the fact that certain physical quantities must be independent of orientation of axes. Consequently, the equations for these quantities must be expressed in invariant form as well. In particular, use is made of tensor invariants and a fundamental form of invariant equation is postulated. Based upon this postulate and from a knowledge of the variables which enter in the problem, a qualitatively complete form for certain equations can be written at once.

OUTLINE OF PRINCIPLES INVOLVED

We define, following Jeffreys,[1] a Cartesian tensor as a quantity having physical significance and which obeys a certain transformation law. In particular, we are concerned with the second-order tensor which in the three-dimensional physical space contains nine elements, usually arranged in the 3 by 3 matric form.

This definition means that for our purposes the tensor is identical with the dyadic of the Gibbs vector notation and in addition, is identical with the linear vector function which has been defined[2] as a quantity which operates on a vector to give a vector.

There are certain tensors which are fundamental to and occur repeatedly in the various fields of applied mechanics. These tensors will be discussed later. For the present we consider certain properties of a typical tensor A, or in matric notation

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The invariants of a three-dimensional tensor are three in number,[3] namely

(1) the trace, or sum of the diagonal elements

$$I_1=a_{11}+a_{22}+a_{33};$$

[1] Harold Jeffreys, *Cartesian Tensors* (Cambridge University Press, London, 1931).

[2] Francis D. Murnaghan, *Introduction to Applied Mathematics* (John Wiley and Sons, Inc., New York, 1948), p. 44.

[3] See reference 2, p. 55.

(2) the sum of the two-rowed principal minors

$$I_2=\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}+\begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}+\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix};$$

(3) the determinant of the matrix

$$I_3=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}.$$

The two invariants for the two-dimensional tensor correspond to I_1 and I_3.

Because the elements of the physical tensor may be thought of as a measure of the variation of the characteristic quantity in a three-dimensional space, and because the three-dimensional physical space volume is in itself an invariant, the following basic hypothesis is put forth:

> Given a physical invariant which is dependent upon the volume V, then if a characteristic tensor for the problem is known, the following must be a form of equation among the variables
>
> $$d\eta/dV=c_1k_1I_1+k_2(c_{21}I_1^2+c_{22}I_2)+k_3(c_{13}I_1^3+c_{23}I_1I_2+c_{33}I_3)+\cdots, \quad (1a)$$
>
> in which η is the invariant, the k's are physical invariants, if any exist, of proper dimensions to give dimensional homogeneity, the c's are constants, and the I's are the tensor invariants.

There is not necessarily only one equation of this form for any given problem, since all physical problems have more than one tensor. Indeed, there may even be other forms of the equation which do not use tensor invariants. But if the tensor elements appear in the equation, they must appear in the above form.

The general three-dimensional surface area element is a vector quantity. Therefore, its elements are not invariants. An invariant form using surface elements may be obtained by taking the scalar product of $d\mathbf{A}$ with a vector quantity. This is not within the scope of the material covered here. That is, we assume that surface effects do not enter into the equation for the invariant. However, the specialized two-dimensional form of the three-dimensional equations corresponding to identical conditions in parallel planes does apply and in this case the equation takes the form

$$d\eta/dA=c_1k_1I_1+k_2(c_{21}I_1^2+c_{22}I_2)+\cdots. \quad (1b)$$

In Eqs. (1a) and (1b) it will be noted that the form assumed assigns positive integral values to the exponents of the invariants. The reason for this is as follows. Any of the tensor invariant quantities may be either positive, negative, or zero. The left-hand side of the equation is made up of real quantities. In order that the right-hand side may not have imaginary or infinite terms the exponents are chosen as positive integers. Some illustrations of the applications of these equations will be cited from the various fields of applied mechanics.

PLATES AND SHELLS

In this case use may be made of results obtained for the analogous one-dimensional structure, the beam. Thus we find the fundamental quantities that enter into beam analysis are d^2w/dx^2 or $1/r$, the curvature; M, the bending moment; $d^2(d^2w/dx^2)/dx^2$; d^2M/dx^2; and EI the stiffness.

By analogy, the corresponding tensor quantities for the two-dimensional thin plate are given by the four expressions following.

$$\begin{pmatrix}\dfrac{\partial^2 w}{\partial x^2} & \dfrac{\partial^2 w}{\partial x\partial y}\\[2ex] \dfrac{\partial^2 w}{\partial y\partial x} & \dfrac{\partial^2 w}{\partial y^2}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\dfrac{1}{r_x} & \dfrac{1}{r_{xy}}\\[2ex] \dfrac{1}{r_{yx}} & \dfrac{1}{r_y}\end{pmatrix}, \quad (2a)$$

$$\begin{pmatrix}M_x & -M_{xy}\\ M_{yx} & M_y\end{pmatrix}, \quad (2b)$$

$$\begin{pmatrix}\dfrac{\partial^2}{\partial x^2} & \dfrac{\partial^2}{\partial x\partial y}\\[2ex] \dfrac{\partial^2}{\partial y\partial x} & \dfrac{\partial^2}{\partial y^2}\end{pmatrix}\cdot\begin{pmatrix}\dfrac{\partial^2 w}{\partial x^2} & \dfrac{\partial^2 w}{\partial x\partial y}\\[2ex] \dfrac{\partial^2 w}{\partial y\partial x} & \dfrac{\partial^2 w}{\partial y^2}\end{pmatrix}$$

$$=\begin{pmatrix}\dfrac{\partial^4 w}{\partial x^4}+\dfrac{\partial^4 w}{\partial x^2\partial y^2} & \dfrac{\partial^4 w}{\partial x^3\partial y}+\dfrac{\partial^4 w}{\partial x\partial y^3}\\[2ex] \dfrac{\partial^4 w}{\partial x^3\partial y}+\dfrac{\partial^4 w}{\partial y^3\partial x} & \dfrac{\partial^4 w}{\partial y^4}+\dfrac{\partial^4 w}{dx^2dy^2}\end{pmatrix}, \quad (2c)$$

$$\begin{pmatrix} \dfrac{\partial^2}{\partial x^2} & \dfrac{\partial^2}{\partial x \partial y} \\ \dfrac{\partial^2}{\partial y \partial x} & \dfrac{\partial^2}{\partial y^2} \end{pmatrix} \cdot \begin{pmatrix} M_x & -M_{xy} \\ M_{yx} & M_y \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial^2 M_x}{\partial x^2}+\dfrac{\partial^2 M_{yx}}{\partial x \partial y} & -\dfrac{\partial^2 M_{xy}}{\partial x^2}+\dfrac{\partial^2 M_y}{\partial x \partial y} \\ \dfrac{\partial^2 M_x}{\partial x \partial y}+\dfrac{\partial^2 M_{yx}}{\partial y^2} & -\dfrac{\partial^2 M_{xy}}{\partial x \partial y}+\dfrac{\partial^2 M_y}{\partial y^2} \end{pmatrix}. \quad (2d)$$

In Eq. (2b) the negative sign is chosen to conform to the usual sign convention. Corresponding to the stiffness we have the so-called flexural rigidity, D, of the plate. The tensors (2c) and (2d) are a measure of the stiffness of plates, that is the ability of the plate to resist bending as well as membrane stresses; and they represent a matric multiplication.

Now, the energy stored in a beam subjected to bending moments is

$$\eta = (EI/2)\int_l (d^2w/dx^2)^2 dx. \quad (3a)$$

For the plate, by analogy, we would look for an invariant expression of the form

$$\eta = D\int_A \left[c_{21}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 + c_{22}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{dy^2} - \left\{\frac{\partial^2 w}{\partial x \partial y}\right\}^2\right)\right] dA, \quad (3b)$$

and this is the actual form as given by Timoshenko.[4]

Other possible forms of this follow at once as

$$\eta = D\int_A \left[c_{21}\left(\frac{1}{r_x}+\frac{1}{r_y}\right)^2 + c_{22}\left(\frac{1}{r_x}\frac{1}{r_y} - \left\{\frac{1}{r_{xy}}\right\}^2\right)\right] dA, \quad (3c)$$

$$\eta = (1/D)\textstyle\int_A [c_{21}'(M_x+M_y)^2 + c_{22}'(M_xM_y - M_{xy}^2)]dA. \quad (3d)$$

As another example consider a plate subjected to a transverse load of q pounds per unit area. This loading cannot depend upon orientation of the axes and in addition can be thought of as a term of the proper derivative form $d(\ )/dA$. Referring again to the beam problem, we have for beams with q equal to the running load per unit length

$$q = d^2M/dx^2 = EId^2(d^2w/dx^2)/dx^2 \quad (4a)$$

The extension to plates gives[5]

$$q = c_1\left(\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_{yx}}{\partial x \partial y}+\frac{\partial^2 M_y}{\partial y^2}-\frac{\partial^2 M_{xy}}{\partial x \partial y}\right), \quad (4b)$$

and

$$q = Dc_1'\left(\frac{\partial^4 w}{\partial x^4}+\frac{2\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right). \quad (4c)$$

ELASTICITY

The tensors which appear in isotropic elasticity are the stress and strain tensors given respectively by

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

and

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{bmatrix}.$$

Just as before, some information may be obtained by analogy to the simpler types of elastic action. Thus, the strain energy stored in an isotropic body subjected to pure tension is

$$\eta = (E/2)\int_V (\partial u/\partial x)^2 dV. \quad (5a)$$

This leads at once to the following general expression for strain energy

$$\eta = E\int_V \left[c_{21}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2 + c_{22}\left(\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial v}{\partial y}-\frac{1}{4}\left\{\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right\}^2 - \frac{1}{4}\left\{\frac{\partial w}{\partial z}+\frac{\partial w}{\partial x}\right\}^2 - \frac{1}{4}\left\{\frac{\partial w}{\partial y}+\frac{\partial v}{dx}\right\}^2\right)\right] dV, \quad (5b)$$

and although this differs in form from the ex-

[4] S. Timoshenko, *Theory of Plates and Shells* (McGraw-Hill Book Company, Inc., New York and London, 1940), p. 50.

[5] See reference 4, pp. 87, 88.

pression given in Love[6] the two equations may easily be shown to be identical.

We consider next the St. Venant torsion problem. Using the notation of Love[7] we have for the strain matrix

$$\begin{bmatrix} 0 & 0 & e_{zx} \\ 0 & 0 & e_{yz} \\ e_{zx} & e_{zy} & 0 \end{bmatrix}, \tag{6a}$$

with

$$e_{zx}=\tau\left(\frac{\partial\phi}{\partial x}-y\right),$$

and

$$e_{zy}=\tau\left(\frac{\partial\phi}{\partial y}+x\right). \tag{6b}$$

The strain tensor elements are in general nondimensional. For this case however, because τ is also an invariant, we have in effect dimensional strains given by the portion in the parentheses. Hence, the invariant for the torsion problem is

$$I=\left(\frac{\partial\phi}{\partial x}-y\right)^2+\left(\frac{\partial\phi}{\partial y}+x\right)^2,$$

and for the energy stored in a bar due to torsion we have[7]

$$\eta=c\mu\tau^2\int_V\left[\left(\frac{\partial\phi}{\partial x}-y\right)^2+\left(\frac{\partial\phi}{\partial y}+x\right)^2\right]dV, \tag{7}$$

μ being the invariant modulus of elasticity in shear.

Also, the applied moment M_z is certainly independent of the orientation of the x and y axes, and consequently an equation of the following form is looked for

$$M_z=c\mu\tau\int_A\left[\left(\frac{\partial\phi}{\partial x}-y\right)^2+\left(\frac{\partial\phi}{\partial y}+x\right)^2\right]dA. \tag{8}$$

and this may easily be transformed into the equivalent expression given by Love.[7]

CAPILLARITY, MEMBRANES, AND VIBRATIONS

The problems in these fields are somewhat similar to those in the field of thin plates; hence, in general, the same tensors apply.

In capillarity and membrane problems we have a relation between pressures, edge stresses and radii of curvature. An elementary dimensional and invariant analysis indicates that the form of equation is

$$p=sc\left(\frac{1}{r_x}+\frac{1}{r_y}\right), \tag{9a}$$

or, equivalently

$$p=sc\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right), \tag{9b}$$

where s is the edge stress in pounds per unit length.

Finally, in the vibration of plates, the inertia term $k\partial^2u/\partial t^2$ corresponds to the q loading of the ordinary plate theory. Hence the extended form of Eq. (4c) becomes

$$k\frac{\partial^2 u}{\partial t^2}=Dc_1\left(\frac{\partial^4 w}{\partial x^4}+\frac{2\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right). \tag{10}$$

FLUID MECHANICS

Some examples from the field of fluid mechanics will now be considered. The problem is rather more involved in these cases, however, because the energy of a fluid can, in general, consist of kinetic, potential, and intrinsic energies. However, some information may still be obtained if we consider an unbounded, steady-state fluid with uniform conditions at infinity, and state the energy equation in terms of the invariants of the rate of strain tensor (which is similar to the strain tensor of elasticity except that u, v, and w are velocities instead of displacements).

Then

$$\begin{aligned} d\eta/dV=&c_1k_1I_1+k_2(c_{21}I_1^2+c_{22}I_2)\\ &+k_3(c_{31}I_1^3+c_{32}I_1I_2+c_{33}I_3)+\cdots. \end{aligned} \tag{11a}$$

Now, if the fluid is incompressible, $I_1=0$ by virtue of continuity, and k_2 must have the dimensions of viscosity for dimensional homogeneity. There is no physical invariant k_3 of the proper dimensions. Consequently, if a fluid is nonviscous and incompressible it would seem that the mechanism for dissipating energy is not present and hence drag on the body is not pos-

[6] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Cambridge University Press, London, third edition, 1920), p. 166.

[7] See reference 6, pp. 316, 317, and 318.

sible—which is a statement of d'Alembert's paradox.

If the fluid is viscous and incompressible, then the physical constant, viscosity, permits an expression of the form

$$d\eta/dV = \mu(c_{21}I_1^2 + c_{22}I_2), \quad (11b)$$

which is equivalent to the energy dissipation equation as given by Lamb.[8]

If the flow is compressible and nonviscous, then $I_1 \neq 0$ and a permissible energy dissipation equation takes the form

$$\frac{d\eta}{dV} = cp\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \quad (11c)$$

[8] Horace Lamb, *Hydrodynamics* (Dover Publications, New York, 1945), sixth edition, p. 580.

The right-hand side of the above, as shown by Lamb,[9] represents the rate of dissipation of intrinsic energy.

CONCLUSION

A method of dimensional reasoning has been discussed which is based upon the invariants of tensors and a postulated form of certain equations in applied mechanics. On the basis of this procedure it is shown that the differential forms of some of the invariant equations in mechanics may be obtained, qualitatively, by consideration of the characteristic tensors of the fields in question.

[9] See reference 8, p. 9.

On the Representation of the Static Polarization of Rigid Dielectrics by Equivalent Charge Distributions

L. H. FISHER
New York University, University Heights, New York

(Received July 24, 1950)

The representation of statically polarized rigid dielectrics (homogeneous isotropic, homogeneous anisotropic, nonhomogeneous isotropic) by equivalent induced surface and volume charge distributions is examined in detail. Attempts at interpretation of equivalent surface charge densities in crystals (isotropic and nonisotropic) and of equivalent volume charge densities in crystals (nonisotropic) indicate that, for these cases, the concepts are artificial and are of limited physical significance. Difficulties in interpretation do not arise for gases, liquids, amorphous solids, or solids made up of large numbers of randomly oriented crystals. The corresponding magnetostatic problems are discussed, although interpretation is not attempted in view of the artificiality of the concept of the magnetic pole.

THE fundamental laws governing the macroscopic phenomena of electrostatics are Maxwell's first equation[1]

$$\nabla \cdot \mathbf{D} = 4\pi\rho \quad (1)$$

and the irrotationality of the electric field $\mathbf{E}$. Here ρ is the volume charge density of true charge (the charge density which would exist if the dielectric were removed), and $\mathbf{D}$ is the electric displacement. In vacuum $\mathbf{D}$ can be replaced by $\mathbf{E}$.

The general definition of the polarization vector $\mathbf{P}$ is given by the equation

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}. \quad (2)$$

[1] Gaussian units are used throughout.

Taking the divergence of Eq. (2), and then inserting Eq. (1), one obtains

$$4\pi\rho = \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{E} + 4\pi\nabla \cdot \mathbf{P}, \quad (3)$$

or

$$\nabla \cdot \mathbf{E} = 4\pi(\rho - \nabla \cdot \mathbf{P}). \quad (4)$$

It is seen that if the dielectric were removed, an additional volume charge density ρ' would have to be assigned to every point in space in order to give the same electric field in vacuum as is observed in the presence of the dielectric. From Eq. (4) it is seen that where $\nabla \cdot \mathbf{P}$ is finite

$$\rho' = -\nabla \cdot \mathbf{P}. \quad (5)$$

Where $\nabla \cdot \mathbf{P}$ is not finite, the "surface divergence" must be included by applying Gauss' theorem to

a small volume at the discontinuity. There is then assigned in addition to ρ' an induced surface charge density ω', where

$$\omega' = (\mathbf{P}_1 - \mathbf{P}_2) \cdot \mathbf{n}. \quad (6)$$

Here $\mathbf{n}$ is a unit vector normal to the surface of discontinuity, the positive direction being taken from medium 1 to 2. Likewise $\mathbf{P}_1$ and $\mathbf{P}_2$ are the polarizations at the two surfaces of the boundary. Only boundaries which separate dielectric from vacuum will be considered. Hence, one of the polarization vectors ($\mathbf{P}_2$) vanishes, and

$$\omega' = \mathbf{P} \cdot \mathbf{n}, \quad (7)$$

where the subscript 1 has been dropped, and $\mathbf{n}$ points from the dielectric into the vacuum. Actually, sharp discontinuities in macroscopic physical quantities do not exist. However, such physical quantities may have very large space derivatives over short distances, and this situation may be treated mathematically by assuming the existence of abrupt discontinuities. Hence, there is no fundamental difference between induced surface and volume charge densities. Whether distributions of true or induced charges are being discussed, it is more meaningful physically to speak of volume charge density than of surface charge density. However, where the charges are confined to a volume whose thickness is small compared to the other dimensions of the volume, it is more convenient to treat the charge distribution as a surface charge, and the usual distinction between surface and volume charge densities will be made in this paper.

The total charge Q' induced in the dielectric is given by

$$Q' = -\int \nabla \cdot \mathbf{P} d\tau + \int \mathbf{P} \cdot d\boldsymbol{\sigma}, \quad (8)$$

where $d\tau$ and $d\sigma$ are volume and surface elements, respectively, and it is seen from Gauss' theorem that $Q' = 0$, in general.

Various authorities have expressed themselves on the nature of this induced charge, and we give here an example of three points of view.

Maxwell[2] states: "The apparent charge of electricity within a given region may increase or diminish without any passage of electricity through the bounding surface of the region. We must therefore distinguish it from the true charge, which satisfies the equation of continuity." On the other hand Richardson[3] states: "Although it is necessary, in discussing the results of electrostatic experiments, to distinguish between 'true' charges like those which are communicated from a conductor to the plates of a condenser and the 'fictitious' charges which appear to reside in the dielectric, there is no very profound difference between them. According to the electron theory one is just as true a charge as the other, although its reality is not so readily made obvious by experiment." Still a third point of view is expressed by Houston[4] who writes: "The fact that the divergence of the polarization and the normal component of the polarization··· are equivalent to volume and surface charge densities has led to numerous attempts to visualize the effect in terms of the charges composing the dipoles···Such visualization clearly violates restrictions imposed on the size of the volume elements that can be used and must be employed with caution. In a similar fashion the ρ (our Eq. (1)) is sometimes referred to as a 'real' charge density while the $-\nabla \cdot \mathbf{P}$ is called a 'bound' charge density. This also seems of doubtful value, and it appears more practicable to recognize that the integral $\int(\rho - \nabla \cdot \mathbf{P})d\tau/r$ is just a method of calculating the potential due to both charges and dipoles."

One certainly agrees with Maxwell that the induced charge must be a "different" kind of charge of little physical interest if it does not obey a conservation law. On the other hand, it has already been seen that the conservation law holds if the entire dielectric is included. Consider the case where the volume does not include the entire dielectric. Then[5] the polarization charge is conserved by assigning to each element of the arbitrary surface surrounding the volume a surface charge density of $\mathbf{P} \cdot \mathbf{n}$. On the same surface, but assigned to the remaining volume of the dielectric, there should be assigned an equal and opposite surface charge density. This assignment

[2] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Third Ed. (Oxford University Press, London, 1892), vol. I, p. 100.

[3] O. W. Richardson, *The Electron Theory of Matter* (Cambridge University Press, London, 1914), p. 57.

[4] W. V. Houston, *Principles of Mathematical Physics*, Second Ed. (McGraw-Hill Book Company, Inc., New York, 1948), p. 260.

[5] See reference 3, p. 49.

is reasonable from our present point of view, for if an infinitesimal cut were made between the two surfaces, equal and opposite induced surface charge densities of magnitude $\mathbf{P}\cdot\mathbf{n}$ are normally assigned. Thus, the conservation law can be made to hold for any volume. Maxwell's reference to induced charge as fictitious charge on the basis of the absence of a conservation law is not wholly justified. The situation, however, is not completely clarified by Richardson's statement, and the difficulties which arise will be discussed later. It is the purpose of this paper to emphasize and amplify the point of view of Houston.

At such points where $\rho=0$, it follows from Eq. (3) that

$$\rho'=\nabla\cdot\mathbf{E}/4\pi. \tag{9}$$

Equations (1) through (9) are independent of the nature of the dielectric, and will now be applied to several cases.

Case I. Homogeneous Isotropic Dielectrics

This is the most familiar case. That class of dielectrics is discussed here for which $\mathbf{P}$ is proportional to $\mathbf{E}$ and to $\mathbf{D}$. Hence $\nabla\cdot\mathbf{P}$ is proportional to $\nabla\cdot\mathbf{D}$. Hence $\rho'=0$ everywhere except at such points where there is true charge.[6] This result does not hold at surfaces of discontinuities and at such surfaces induced charges appear.

If at this point of the development, one tries to interpret these induced volume charge densities physically, because of the particle nature of electricity, it would seem best to consider ρ' not as a volume charge density, but rather as a manifestation of a surface charge. These surface charges are induced on the surfaces of the small cavities in which the true charges are placed.[7] At all other points in the dielectric where $\rho=0$, ρ' is also zero. Whether one accepts the interpretation of ρ' as a surface charge density or not, and regardless of the physical interpretation of induced charge densities in general, for a homogeneous isotropic dielectric, regardless of the shape of the dielectric, it is impossible according to Eq. (9) to produce a volume charge density of induced charge in any element of the dielectric by any charge distribution outside the volume element. The result just stated depends explicitly on the proportionality between $\mathbf{P}$ and $\mathbf{D}$, and on the inverse square law of force ($\nabla\cdot\mathbf{r}/r^3=0$).

The concept of induced surface charges is generally used to give insight into the reduction of forces between true charges when the charges are immersed in a homogeneous isotropic medium. It will be shown later that this point of view is not satisfactory for crystals. The fact that for this important case (Case I), ρ' is, in general, zero has led to a neglect of the physical interpretation of the induced volume charge density. In Case II, it is found that this question must be discussed in detail. After a discussion of the induced volume charge density, the question of induced surface charge density will be studied further.

Case II. Homogeneous Anisotropic Dielectrics

This case is of considerable interest. Only the simplest soluble problem is presented, but the conclusions obtained apply quite generally. Consider an infinite homogeneous anisotropic crystalline dielectric for which the principal axes of the dielectric tensor lie in the x, y, and z directions. Let the values of the dielectric tensor in these three directions be K_x, K_y, and K_z. Then the relationship between $\mathbf{D}$ and $\mathbf{E}$ is given by

$$D_i=K_iE_i,\quad i=x,y,z, \tag{10}$$

where all of the K_i are constants, and at least two of the K_i are different. Consider the case where $\rho=0$ and Eq. (9) applies. To evaluate ρ' at such a point, let us assume that at some other region in the dielectric an ellipsoidal cavity is made and that a true charge Q on an ellipsoidal conductor is placed in the cavity. Let the conductor and the cavity have the same dimensions with the principal axes of the conductor lying in the x, y, z directions; let the length of the principal axes be proportional to $K_x^{\frac{1}{2}}$, $K_y^{\frac{1}{2}}$, $K_z^{\frac{1}{2}}$ in the x, y, and z directions, respectively. The potential φ everywhere except at the surface of the con-

[6] The result that $\rho'=0$ for $\rho=0$ in a homogeneous isotropic medium is given in several places, see for example L. Page, *Introduction to Theoretical Physics*, Second Ed. (D. Van Nostrand Company, Inc., New York, 1935), p. 375 or M. Mason and W. Weaver, *The Electromagnetic Field* (The University of Chicago Press, Chicago, 1929), p. 143.

[7] The case where the true charge density varies continuously with position is not considered as it is of little physical interest.

ductor (and inside the conductor) is described by

$$\sum_i K_i \partial^2\varphi/\partial i^2 = 0, \quad i = x, y, z. \tag{11}$$

Let the coordinates in Eq. (11) be transformed so that

$$i = K_i^{\frac{1}{2}} i', \quad i = x, y, z, \tag{12}$$

where the coordinates are measured from the center of the cavity. In terms of the new coordinates, one obtains

$$\Delta'\varphi = 0, \tag{13}$$

where the prime indicates that the differentiations are to be carried out with respect to the primed coordinates. A solution of Eq. (13) which can be made to satisfy the boundary conditions is of the form $1/r'$, where $r'^2 = \sum_i i'^2$. In terms of the unprimed coordinates, one obtains

$$\varphi = AQ\{\sum_i (i^2/K_i)\}^{-\frac{1}{2}}, \quad i = x, y, z. \tag{14}$$

Here A is a constant which has the "dimensions" of the dielectric constant raised to the negative three-halves power. The components of the electric field are described by

$$E_j = (jAQ/K)\{\sum_i (i^2/K_i)\}^{-\frac{3}{2}}, \quad i = x, y, z, \tag{15}$$

where j takes on successively the values x, y, z. Equations (14) and (15) for the potential and field satisfy the following conditions: (a) $\mathbf{E}$ is irrotational, (b) the potential of the conducting ellipsoid is a constant, and (c) $\mathbf{D}$ is solenoidal everywhere except at the surface of the conductor. All that remains for the complete solution is the evaluation of the constant A, and for the purpose of this discussion, this need not be carried out. It is interesting to note in passing, that $\mathbf{D}$ is radial with respect to the center of the cavity, but has different magnitudes in different directions. The electric field is everywhere at right angles to the ellipsoidal equipotential surfaces.

Now one is able to calculate ρ' at a point where $\rho = 0$ by Eq. (9). Carrying out the indicated operations, one obtains

$$\rho' = (AQ/4\pi)[(\sum_i K_i^{-1}) - 3\{\sum_i (i^2/K_i^2)\} \times \{\sum_i (i^2/K_i)\}^{-1}][\sum_i (i^2/K_i)]^{-\frac{3}{2}}, \quad i = x, y, z. \tag{16}$$

In general, Eq. (16) will not give $\rho' = 0$ (unless $K_x = K_y = K_z$) and one expects induced volume charges in homogeneous anisotropic media.[8]

One may now inquire into the physical meaning of this induced volume charge density. As is well known, it can be shown quite generally[9] that a *continuous* distribution of dipoles is equivalent to a volume and surface charge distribution. Consider an arbitrary volume of the polarized dielectric containing many atoms. Suppose the mean distance of the negative from the positive charge in a single atomic dipole is small compared to the distance between the atoms.[10] This will in general be the case. Then one can obtain values of $\rho' = 0$ if one includes an integral number of elementary dipoles in the volume, or one can by stretching or contracting the volume by a small amount include only the positive charges of the dipoles on the surface. Obviously, a charge density which depends so critically on the choice of the volume element needs a more detailed discussion. In the almost impossible case (electric breakdown will take place first) that the separation of negative and positive charge in a single atom is large compared to the distance between atoms, then regardless of how the volume element is taken, one obtains a value of ρ' which varies gradually with minute changes in the boundary. However, since the assumption of charge separation large compared to interatomic separation is not justified, it seems that the concept of induced volume charge density, when it does appear formally must be examined.

It is interesting to note that Lorentz[11] in discussing the problem of deriving the macroscopic field equations from the microscopic field equa-

[8] This statement does not depend on the highly specialized problem solved above. Actually, the potential in the dielectric due to any charge distribution in a small cavity will be given by a series of descending spherical harmonics in the primed space; the solution given above is only the $1/r'$ term. In the *Magnetostatic Problem* section, it is shown that the statement also holds for the dipole term; undoubtedly the statement holds for all terms in the expansion.

[9] M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Blackie and Son Ltd., London, 1937), p. 72.

[10] It is assumed in the discussion that the atoms (or molecules) have no dipole moment in the absence of an electric field, but the discussion may be easily extended for the more general case of permanent dipole moments.

[11] H. A. Lorentz, "The fundamental equations for electromagnetic phenomena in ponderable bodies, deduced from the theory of electrons," *Collected Papers* (Martinus Nijhoff, The Hague, 1936), vol. III, p. 122; originally published in Proc. Acad. Amsterdam **5**, 254 (1902).

tions found difficulty in treating the problem of a substance whose atoms are arranged in a periodic way. In such cases, Lorentz found it necessary to introduce "irregular undulations" in the surface of the volume elements. The Lorentz theory of averaging the microscopic field equations over a large number of atoms to give the macroscopic field equations is not in itself enough, but suitable undulations in the volume element must be introduced as well. These undulations may be equivalent to an additional averaging process over all possible angular orientations of an arbitrary physically "infinitely small" volume element.[12,13] At best, one sees that ρ' has a rather artificial meaning in crystals.

The meaning of an induced surface charge density in crystals is just as obscure as the induced volume charge density. Thus the common practice of regarding electrified homogeneous isotropic crystalline media in terms of induced surface charges does not seem meaningful. Thus if one wishes to discuss the reduction of the force between two charges when placed in an infinite homogeneous medium, it is not too significant to do so on the basis of the effects of the surface charges on the cavities containing the charges. The forces should be discussed on the basis of all of the isolated dipoles in the dielectric.

For the case of gases, liquids,[14] amorphous solids, or solids made up of large numbers of randomly oriented microscopic crystals, there is no difficulty in interpreting induced surface charge densities (according to Case I, $\rho'=0$ for these substances).

Case III. Nonhomogeneous Isotropic Dielectrics

This case is of little physical interest, introduces no new ideas, but is included for completeness. As in Cases I and II, the effect of electrostriction is neglected. Here $\mathbf{D}=K\mathbf{E}$, where K is a scalar function of position. Here again, for a point at which there is no true charge, Eq. (9) holds. Also

$$\nabla\cdot\mathbf{D}=\nabla\cdot(K\mathbf{E})=K\nabla\cdot\mathbf{E}+\mathbf{E}\cdot\nabla K. \tag{17}$$

Thus ρ' will again in general be different from zero at points where $\rho=0$. Here, however, the medium may be considered to be composed of a large number of small volume elements with different dielectric constants. The induced volume charge density is then an expression of the induced surface charges between the media of different dielectric constants. Hence, the previous discussion of induced surface charge covers this case as well.

The Magnetostatic Case

The macroscopic phenomena of magnetostatics are described by the equation

$$\nabla\cdot\mathbf{B}=0, \tag{18}$$

and the condition that the magnetic field $\mathbf{H}$ is irrotational. Here $\mathbf{B}$ is the magnetic induction. The intensity of magnetization $\mathbf{M}$ is defined by

$$\mathbf{B}=\mathbf{H}+4\pi\mathbf{M}. \tag{19}$$

This leads to

$$\nabla\cdot\mathbf{H}=-4\pi\nabla\cdot\mathbf{M}, \tag{20}$$

and one sees that

$$\rho_m=-\nabla\cdot\mathbf{M}, \tag{21}$$

or,

$$\rho_m=\nabla\cdot\mathbf{H}/4\pi, \tag{22}$$

where ρ_m is the density of magnetic poles.

In media for which $\mathbf{B}$ is proportional to $\mathbf{H}$ (constant permeability), one obtains results similar to Case I for electrostatics. Here one places permanent magnets in cavities in the medium. For the case of constant permeability, $\rho_m=0$, and the induced magnetic charges are distributed only on the boundaries of the medium as magnetic poles.

The case of an anisotropic crystal in magnetostatics may be discussed in a way analogous to Case II of electrostatics, since the permeability tensor is symmetric. Again a simple problem is chosen for solution, but the conclusions are quite general. The problem which can be solved conveniently is the one for which a special ellipsoidal permanent magnet is placed in an ellipsoidal cavity in the medium. Let μ_x, μ_y, μ_z be the values of the components of the permeability tensor along the x, y, and z directions, and let the magnitude of the axes in these directions be proportional to $\mu_x^{\frac{1}{2}}$, $\mu_y^{\frac{1}{2}}$, $\mu_z^{\frac{1}{2}}$, respectively. Let the permanent magnet have the same dimensions as the cavity. Let the magnet be magnetized

[12] A physically "infinitely small" volume element contains an enormous number of atoms, but is so small that the derivatives of the macroscopic fields are constant over the extent of the volume.

[13] A similar suggestion for averaging polarization currents over arbitrarily oriented surfaces in crystals has been made by R. Becker, *Theorie der Elektrizität* (B. G. Teubner, Leipzig, 1933), vol. II, p. 114.

[14] The case of liquid crystals is omitted from the discussion.

so that one can consider it to have a magnetic moment per unit area obtained as follows. Consider an ellipsoid of the above dimensions with magnetic pole density of south poles which is proportional to the actual electric surface charge density on the conducting ellipsoid of Case II, if the dielectric tensor had components μ_x, μ_y, μ_z. Now consider a second identical ellipsoid, except that the magnetic surface pole density is of the opposite sign. The first ellipsoid has its center at (0, 0, 0) and the second ellipsoid has its center at $(c, 0, 0)$. Let the respective axes of the two magnetic ellipsoids be parallel. Now let c approach zero such that, in the usual way, the product of magnetic pole density and the distance c remains fixed for any given point on the ellipsoid. This ellipsoid, magnetized along the x axis is the permanent magnet placed in the cavity.

The magnetic potential φ_m in the medium due to only one of the two superimposed ellipsoids is given by

$$\varphi_m = C/r' \tag{23}$$

where $r'^2 = \sum_i i'^2$, $(i' = x', y', z')$, and $i = \mu_i^{\frac{1}{2}} i'$, $(i = x, y, z)$, and C is a constant. The magnetic potential φ_m' due to both ellipsoids is obtained by differentiating Eq. (23) with respect to x' and multiplying the result by a constant. One thus obtains

$$\varphi_m' = Fx'/r'^3, \tag{24}$$

or

$$\varphi_m' = Gx\{\sum_i (i^2/\mu_i)\}^{-\frac{3}{2}}, \tag{25}$$

where F and G are constants. The components of $\mathbf{H}$ in the medium are given by

$$\begin{aligned} H_x &= -G[\{\sum_i (i^2/\mu_i)\}^{-3/2} \\ &\qquad - (3x^2/\mu_x)\{\sum_i (i^2/\mu_i)\}^{-5/2}], \\ H_y &= (3Gxy/\mu_y)\{\sum_i (i^2/\mu_i)\}^{-5/2}, \\ H_z &= (3Gxz/\mu_z)\{\sum_i (i^2/\mu_i)\}^{-5/2}. \end{aligned} \tag{26}$$

One can now evaluate ρ_m by applying Eq. (22) to Eq. (26), and one obtains

$$\begin{aligned} \rho_m = (3Gx/4\pi)[&\{\sum_i (i^2/\mu_i)\}^{-5/2}(3\mu_x^{-1} \\ &+\mu_y^{-1}+\mu_z^{-1}) - 5\{\sum_i (i^2/\mu_i)\}^{-7/2} \\ &\times \{\sum_i (i^2/\mu_i^2)\}]. \end{aligned} \tag{27}$$

Equation (27) will in general not give $\rho_m = 0$, and one sees that the difficulties encountered in the analogous case in electrostatics appear here as well. Here, however, one has to deal with the fundamentally artificial concept of a magnetic pole, and no interpretation is attempted.

Case III for magnetostatics need not be discussed in detail as the extension from the electrostatic case is clear. However, an additional observation may be made for the case of permanent magnets. If a large number of permanent saturated ideal magnets are connected, one can obtain ρ_m different from zero by suitably varying the intensity of magnetization of the magnets. In this case, however, ρ_m can be interpreted as surface pole charges on the surfaces between the media of different intensities of magnetization.

Summary

The usual representation of a statically polarized rigid dielectric by an equivalent induced surface and volume charge distribution has been examined. Three types of dielectrics have been studied in detail: Case I, homogeneous isotropic; Case II, homogeneous anisotropic; Case III, nonhomogeneous isotropic.

Induced volume charge densities need never appear formally in Case I. In Case II, (Case III can be reduced to Case I) induced volume charge densities do appear, and it is difficult to interpret these densities. Surface charge densities are equally difficult to interpret for crystals. These difficulties do not appear for materials in which the atoms of the dielectric are arranged in a random way (gases, liquids, amorphous solids, and solids made up of large numbers of randomly oriented microscopic crystals).

The analogous magnetostatic cases have been discussed, and it has been shown that the difficulties encountered in the electrostatic case are also met here. The interpretation of magnetic pole densities has not been attempted in view of the artificial nature of the concept of the magnetic pole.

The author is indebted to Professor F. Reiche and Mr. M. Menes for several stimulating discussions.

World Trends in the Publication of Physical Research, 1938–1948

JOHN J. MCCARTHY
Newark College, Rutgers University, Newark, New Jersey
(Received May 11, 1950)

This study is based on the abstracts published in *Physics Abstracts* (*Science Abstracts, Section A*) in 1938 and 1948. In 1938 the twelve countries credited with the most abstracts ranked as follows: United States, Germany, France, Great Britain, Soviet Union, Japan, Netherlands, Italy, India, Poland, Switzerland, and Canada. In 1948 the standings of the leading twelve were: United States, Great Britain, France, Germany, Netherlands, Soviet Union, India, Italy, Sweden, Canada, Austria, and Switzerland.

India and Japan in 1938, and India alone in 1948, were the only countries outside of Europe and North America with an appreciable number of abstracts. Less than half of the countries in the world publish the results of physical research.

The emigration of physicists and economic conditions may account for the decline of published articles in many countries.

THE great interest in physics following World War II has given rise to the belief that research in all branches of physics has increased by leaps and bounds. While this article cannot support or deny this assertion, it will compare the publication of physical research in 1938, the last full year before hostilities began, with 1948, when presumably the world had returned to "normal."

The source of data, *Physics Abstracts*, *Section A* of *Science Abstracts*, underwent considerable alteration during the period 1938–1948. One important change was the classification of abstracts under headings other than physics in the 1948 edition. The following branches in the 1938 edition were not listed under physics in 1948 and have been omitted from this article: adsorption, alternating current networks, astronomy and astrophysics, colloids, crystal structure and special properties, electrochemistry, geophysics, mathematical methods and theory of measurements, medical radiology and electrology, photochemistry, photography, and thermochemistry.

The columns of Table I in this article have, in general, the same titles as the branches to which they pertain in the 1948 edition of *Physics Abstracts*, *Section A*. Some consolidations were effected, however. Thus, the abstracts under electricity, magnetism, and x-rays in the 1938 edition were compared with those under electricity, electric currents, electric discharges and radiations, electrodynamics, electromagnetism, and magnetism in the 1948 volume. Likewise, "Structure of Solids" includes not only the abstracts under this heading but also those under elasticity, strength, and rheology.

Since the titles of the columns were based on the 1948 classification, it was necessary to determine how the 1938 abstracts should be apportioned among them. In many cases this did not present much of a problem as the branch titles were the same or nearly the same. More difficult cases were resolved by consulting the "Notes on the Universal Decimal Classification System" in the 1948 edition.

The number of abstracts in capillarity, kinetic theory of liquids, kinetic theory of gases, mechanics of gases, mechanical measurements, heat, thermodynamics, electricity and magnetism, and optics, radiation, and spectra were less in 1948 than in 1938. Table I in columns A through I shows the fields in which there was a decrease in the number of abstracts published. The remaining columns represent fields which in 1948 had the greater number of abstracts published.

In 1938 the twelve countries credited with the most abstracts were in order: United States, Germany, France, Great Britain, Soviet Union, Japan, Netherlands, Italy, India, Poland, Switzerland, and Canada. In 1948 the leading twelve were: United States, Great Britain, France, Germany, Netherlands, Soviet Union, India, Italy, Sweden, Canada, Austria, and Switzerland. Of these only the United States, Great Britain, Austria, and Sweden actually had more abstracts in 1948 than in 1938. The listing of the

TABLE I. Numbers of prewar and postwar publications in various countries in: A, capillarity; B, kinetic theory of liquids; C, kinetic theory of gases; D, mechanics of gases; E, mechanical measurements; F, heat; G, thermodynamics; H, electricity and magnetism; I, optics, radiation, and spectra; J, fundamentals; K, general mechanics and mechanics of solids; L, mechanics of liquids; M, structure of solids; N, vibrations and acoustics; O, atomic and molecular structure; P, cosmic rays; and Q, radioactivity, nuclear particles, and nuclear disintegration. The number on the left in each column refers to the number of abstracts published in 1938; the one on the right to the number published in 1948.

	A		B		C		D		E		F		G		H		I		J		K		L		M		N		O		P		Q	
United States	43	10	26	15	1	4	4	37	14	19	63	48	39	10	176	180	252	234	24	60	4	12	13	28	15	61	54	135	48	130	33	61	94	279
Great Britain	53	9	13	11	1	1	27	16	6	19	27	32	13	5	149	180	97	80	25	48	7	19	12	33	23	61	16	27	30	47	9	20	39	79
France	16	1	4	3	1		17	3	13	1	25	15	4	3	88	73	133	73	22	49	11	6	6	5	9	13	19	16	18	8	10	10	52	36
Germany	32	3	20	4	2	1	11	10	21	8	33	10	17		268	71	193	47	18	10	6	4	8	5	15	4	27	11	22	17	19	6	47	22
Netherlands	8	1	1	2	5		1	7	1	1	14	9	10	5	33	26	16	22	3	2	2	4	7	8	1	10	7	7	5	4	7	2	10	10
Soviet Union	16	1	9				7	2		1	8	6			42	22	50	28	8	5	4	2	3	9	5	12	6	4	4	3	5	2	20	4
India	2	1					4			3	3	2			19	18	36	33	4	4			2			6	6	1	8	2		2		3
Italy			1			1	5	1			3		4		30	5	19	3	7	4	7		3		7	1	2	9	1	4	1	24	10	8
Japan	17		4		2		11		1		4		3		22		48		16		2		3		4		4		3		2		10	
Sweden			1							1		2			1	4	2	6	1	4	1					1		1		5	1			7
Canada	2	1					1	3			5	5	3		7	4	10	1				1			1	1				1	2			12
Austria								1	1		1	4		1	1	5	5	4		3		1			2			2		2	4	1	11	2
Switzerland				1								1		1	8	4	10	1	4	6		1			2	1	5	2	2	2	1		5	5
Eire															1	1	1		1	9													3	
Norway										1		2			1	3		1										1		1				1
Belgium			1					1					1	1			5	3	1	3	1										1		1	1
Brazil																				6														2
Poland	2		1								7				8	2	17		4	5									2		3		1	
Portugal																				3		2		2										
Denmark						1					1	1				1												1						
Rumania							1				1	1			2		2	3											1				1	
Finland																													1					2
Hungary																														2				
Vatican City							1																				1							
Czechoslovakia																3																		
China																						1		1										
Spain																		1																
Uruguay																		1																
Australia																	1																	
Totals	191	27	81	36	12	8	90	81	57	54	195	138	94	26	856	602	897	541	138	221	45	53	57	91	84	171	147	217	145	228	98	128	304	473

countries in the tables is based on their 1948 ranks with the exception of Japan, who was placed ninth. No abstracts were credited to Japan in 1948.

India and Japan in 1938, and India alone in 1948, were the only countries outside of Europe and North America with an appreciable number of abstracts. In making this statement the Soviet Union has been considered a European country inasmuch as its research papers are published in Moscow and Leningrad.

While there are over seventy countries in the world today, less than half publish the results of physical research. More countries—twenty-one, in fact—contributed to "Optics, Radiation, and Spectra" in Table I than to any other branch.

There will be a tendency on the part of many to interpret the data presented in the foregoing table as indicative of the research activity in the various countries. While the saying, "Where there's smoke, there's fire," probably applies to the publication of research, it should not be taken too literally. No effort has been made to determine where the researches were performed or the citizenship of the persons performing them. If this were done, countries with well-established periodicals would suffer, whereas countries with small numbers of abstracts would benefit. This kind of study, for example, might alter China's position considerably.

It is hardly necessary to remind the reader that many discoveries in the period covered by this article have never been published. Reports from the Soviet Union, for example, indicate that in 1948 large numbers of trained personnel were engaged in physical research under government sponsorship. On the basis of published articles, however, this country is decidedly behind the United States and Great Britain. The lifting of restrictions on publication would place her among the leaders.

The emigration of physicists and economic conditions may account for the decline of published articles in many countries. While economic conditions will be remedied in time, the flight of physicists, especially those qualified to direct others, will leave its mark for generations.

Experiments with Doubly Refracting Crystals

MYRON A. JEPPESEN
Bowdoin College, Brunswick, Maine
(Received June 26, 1950)

The birefringence of mica and of quartz is measured by an interference method which is very accurate for thin sections. The orientation and identification of the axes of mica are determined by observation of the specimen in convergent polarized light. These experiments are suitable for the advanced undergraduate laboratory.

ALTHOUGH there are many beautiful demonstrations of effects caused by double refraction of light in crystals, few experiments are described in the literature suitable for the advanced undergraduate laboratory. Good experiments should be quantitative and should require a reasonable amount of laboratory technique. They should help to clarify the subject and should yield results important in themselves. The investigation described below, in which interference effects are used to evaluate optical constants of mica and quartz, is suggested to meet these requirements.

MEASUREMENT OF THE DIFFERENCE BETWEEN THE REFRACTIVE INDICES OF THE SLOW AND THE FAST RAYS

The correct thickness of a quarter-wave plate of mica is different for different samples of mica. Reference books frequently specify the thickness, but the value is quite likely to be wrong for the particular mica available. It is desirable that the experimenter determine the correct thickness for himself. In order to calculate the thickness the difference between the refractive indices of the mica for the slow and the fast rays must be measured. It is difficult to deter-

FIG. 1. Schematic arrangement for measuring birefringence, showing relative positions of source L, polarizer P, mica M, analyzer A, and spectrometer S.

mine this difference with precision by direct measurement of the indices. It is, however, easily and accurately measured from the interference effects to be described. The birefringence of plates of quartz and calcite cut with faces parallel to the optic axis can be measured by the same method.

The plate is placed between crossed polaroid disks in front of the slit of a spectrometer as shown in Fig. 1. A straight filament tungsten lamp is a suitable light source. A constant deviation spectrometer with a calibrated wavelength drum is convenient. The fast (or slow) axis of the doubly refracting plate is set at approximately 45° to the transmission axis of the polarizer. The spectroscope then shows a continuous spectrum crossed by fairly narrow dark bands. For a given material the number of bands observed depends upon the thickness of the plate. If mica of sufficient thickness to produce several bands is unavailable two or three sheets may be clamped together. The corresponding axes of the superimposed sheets must be made parallel. This is accomplished by observation of the interference pattern in convergent polarized light, later to be described.

The wavelength λ at which an interference band occurs is given by[1]

$$n\lambda = \Delta\mu t. \tag{1}$$

In this equation, n is an integer, t is the thickness of the plate, and $\Delta\mu$ is the difference in the value of the index of refraction for the slow and the fast rays. When the analyzer is turned to the parallel position the bands previously observed disappear and are replaced by bands for which n in Eq. (1) is an integer plus $\frac{1}{2}$. Proceeding from long to short wavelengths, the number n increases by unity with each band. For a quartz plate three mm thick, about 30 dark bands are observed across the spectrum for each condition of the analyzer.

TABLE I. Data for mica from a local quarry.

λ	t/λ	n	$\Delta\mu = n\lambda/t$
6205A	1610	8	0.00497
5512	1812	9	0.00497
4961	2016	10	0.00496
6619	1509	7.5	0.00497
5842	1710	8.5	0.00497
5220	1914	9.5	0.00496

In Fig. 2, n is plotted against the ratio of plate thickness to the wavelength of the interference band for a mica plate 0.999 mm thick. The data are summarized in Table I. The first three rows of the table correspond to crossed polaroids in the arrangement of Fig. 1. The polaroids were parallel for the measurements in the last three rows. The absolute values of n, listed in the third column, are obtained by extrapolation of the data to $t/\lambda = 0$. At each wavelength $\Delta\mu$ is calculated from Eq. (1) and listed in column four.

We see that a quarter-wave plate of this kind of mica for sodium light should be 0.0296 mm thick. It is an interesting exercise to make according to calculation half- and full-wave plates of mica for specific wavelengths and to test them in polarized light.

The results for a quartz plate 3.370 mm thick are recorded in Table II. The wavelength at the center of every fifth interference band is given in the first column. The second column gives the ratio of plate thickness to wavelength. The ratio of change in n to change in t/λ is given in the third column. This ratio is an approximation of the birefringence, $\Delta\mu$. That it is greater than $\Delta\mu$ is shown by the following considerations.

Assume a two-term Cauchy formula for the refractive index:

$$\mu_E = A + B/\lambda^2 \tag{2a}$$

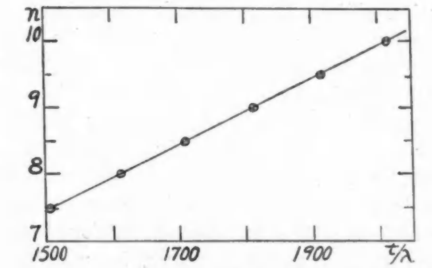

FIG. 2. Number n *versus* t/λ for a mica plate.

[1] The origin of these bands is discussed in most textbooks on physical optics. See, for example, Jenkins and White, *Fundamentals of Physical Optics* (McGraw-Hill Book Company, Inc., New York, 1937), first edition, pp. 362–364.

for the extraordinary ray;

$$\mu_O = A' + B'/\lambda^2 \tag{2b}$$

for the ordinary ray.

Then $\Delta\mu = a + b/\lambda^2$, where $a = A - A'$ and $b = B - B'$.

Substitution of the expression for $\Delta\mu$ into Eq. (1) leads to

$$\Delta n/\Delta(t/\lambda) = a + 3b/\lambda^2. \tag{3}$$

Thus the values listed in the third column of Table II are greater than $\Delta\mu$ by the quantity $2b/\lambda^2$. A precise value of $\Delta\mu$ could possibly be found if the integer n in Eq. (1) were known. The plate is too thick to determine n by extrapolating the data to $t/\lambda = 0$. We can, however, improve on the values listed in the third column of Table II.

Let f_1 be the average of the ratio $\Delta n/\Delta(t/\lambda)$ taken to either side of λ_1, and f_2 the average of the corresponding ratio to either side of λ_2. Then, from Eq. (3),

$$b = (f_2 - f_1)/3(1/\lambda_2^2 - 1/\lambda_1^2). \tag{4}$$

Taking $\lambda_1 = 6208$A, and $\lambda_2 = 4883$A, $f_1 = 0.00994$, and $f_2 = 0.01043$. Substituting in Eq. (4) we get

$$b = 1.02 \times 10^{-10}\ \text{mm}^2.$$

Then $\Delta\mu = f_1 - 2b/\lambda_1^2 = 0.00942$. By Eq. (1), the value of n corresponding to λ_1 is then

$$n_1 = 0.00942 \times 5428 = 51.1.$$

Calculated values of n for other wavelengths are given in the fourth column of Table II. The nearest integer to the computed n is used to evaluate $\Delta\mu$ as recorded in the fifth column. These values are larger than accepted values by 3 or 4 percent.[2]

IDENTIFICATION OF THE AXES OF A MICA PLATE

To use a quarter-wave plate intelligently one must know the directions of the fast and the slow axes. Wood[3] describes two methods by which the axes of a quarter-wave plate may be distinguished. One involves the use of an interferometer, the other the reduction of circularly polarized light of known sense of rotation to plane-polarized light. A very direct method which

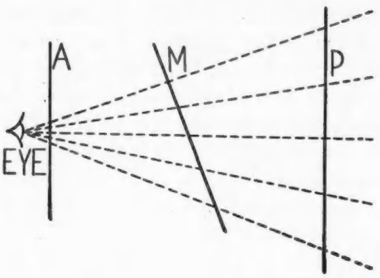

FIG. 3. Schematic arrangement for the identification of the axes of a mica plate, where P, M, A, are the polarizer, mica, and analyzer, respectively.

brings clearly to mind the geometrical relation of the mica plate to its *index ellipsoid* will be described. From this method the angle between the optic axes of the mica is also determined. A cardboard model of the ellipsoid for reference makes the explanation easy to follow.

The equation of the index ellipsoid is

$$x^2/\alpha^2 + y^2/\beta^2 + z^2/\gamma^2 = 1. \tag{5}$$

Here α, β, γ are the principal indices of refraction of the mica, with $\alpha < \beta < \gamma$.

The optic axes of the mica are normal to the two circular cross sections of the ellipsoid. A little reflection (or a glance at a cardboard model) shows that these circular sections intersect along the intermediate axis of the ellipsoid. The optic axes are therefore perpendicular to the intermediate axis, and the plane of the optic axes is the xz plane. If the y-axis is horizontal, rotation about the y-axis causes the optic axes to rotate in a vertical plane.

With the foregoing in mind, imagine the mica (from which a quarter-wave plate is to be split off) between crossed polaroid disks in a beam of

TABLE II. Data for a quartz plate 3.370 mm thick. (Wavelengths are recorded for every fifth band.)

λ	t/λ	$\Delta n/\Delta(t/\lambda)$	n calc.	$\Delta\mu = n\lambda/t$
6845A	4921			0.00935
		0.00986		
6208	5428		51.1	0.00940
		0.01002		
5686	5927		56.2	0.00945
		0.01018		
5251	6418		61.3	0.00951
		0.01035		
4883	6901		66.2	0.00956
		0.01050		
4568	7377			0.00963

[2] *International Critical Tables* (McGraw-Hill Book Company, Inc., New York, 1929), first edition, Vol. 6, p. 341.
[3] Wood, *Physical Optics* (The Macmillan Company, New York, 1934), third edition, pp. 352–354.

convergent light as shown in Fig. 3. The transmission axis of the polarizer P is vertical. The axis of rotation of the mica plate M is perpendicular to the axis of the cone of light. Illumination from a window is adequate if the eye is held close to the analyzer A so that it intercepts a wide cone of light.

When the mica is properly oriented it is rotated about the horizontal axis to such a position that an optic axis coincides with the axis of the light cone intercepted by the eye. This position is recognized by the appearance of an interference pattern, usually described in terms of rings and brushes. The rings are colored interference fringes centered at the point of emergence of the optic axis. When the plane of the optic axes of the mica is parallel to the transmission axis of the polarizer there is a dark vertical brush through the rings. Rotation about the horizontal axis brings first one and then the other of the optic axes into coincidence with the light-cone axis. Having found the proper axis of rotation we have found the direction of the intermediate axis of the index ellipsoid and the vibration direction for rays which travel with a velocity corresponding to the intermediate index β.

As the mica is rotated one finds that the plane of the mica bisects the obtuse angle between the optic axes. A consideration of the index ellipsoid shows that when $\beta-\alpha$ is greater than $\gamma-\beta$ the largest axis of the ellipsoid bisects the obtuse angle between the optic axes. Refractometer measurements show that this is true for mica; hence the largest index corresponds to vibrations in the plane of the mica, perpendicular to the intermediate axis. Likewise the smallest index corresponds to vibrations perpendicular to the plane of the mica. For light passing normally through mica the direction of the intermediate axis of the ellipsoid is therefore the fast axis of the mica plate.

A quarter-wave plate arranged as in Fig. 3 is too thin to show the rings in the interference pattern. However, when it is properly oriented the vertical brush is observed. The fast axis of the plate is then horizontal. This test is very rapid as the mica can be rotated in the fingers to find the fast axis.

To measure the principal indices of the mica with an Abbé type refractometer the light should enter the mica through its edge, which is impossible because the edges are unpolished. Fairly accurate values are obtainable by using a thin sheet of mica surrounded by naphthalene monobromide. Values thus obtained for local mica are:

$$\alpha=1.563, \quad \beta=1.594, \quad \gamma=1.600.$$

In order of magnitude $\gamma-\beta$ agrees with the more precise values listed in Table I. A further check on the results is obtained by the calculation of $\beta-\alpha$ from interference measurements. First the angle $2V$ between the optic axes is calculated from the measured angle $2E$ between the rays in air that have traversed the mica in the directions of the optic axes. By measurement, with the mica in the arrangement of Fig. 3, $2E=70.4°$. By Snell's law, $\beta \sin V=\sin E$. Hence $V=21.2°$.

Now the difference between the intermediate and the smallest index is computed. From the index ellipsoid it can be shown that[4]

$$\tan^2 V=\alpha^2/\gamma^2\{(\gamma^2-\beta^2)/(\beta^2-\alpha^2)\}.$$

To a good approximation this reduces to

$$\tan^2 V=(\gamma-\beta)/(\beta-\alpha).$$

Substituting for $\gamma-\beta$ from Table I we get $\beta-\alpha=0.033$. This value should be more accurate than the difference obtained from direct refractometer measurements.

[4] Preston, *The Theory of Light* (The Macmillan Company, London, 1901), third edition, p. 339.

Every physical theory which survives goes through three stages. In the first stage, it is a matter of controversy among specialists; in the second stage, the specialists are agreed that it is the theory which best fits the available evidence, though it may well hereafter be found incompatible with new evidence; in the third stage, it is thought very unlikely that any new evidence will do more than somewhat modify it.—Bertrand Russell, Human Knowledge, 1948.

Analysis of an *R-C* Oscillator

Joseph H. Howey
Georgia Institute of Technology, Atlanta, Georgia
(Received August 19, 1950)

The equations for an *R-C* oscillator are derived completely from fundamental relationships without the use of alternating current concepts and without the use of complex numbers. Analyzed in this way, the *R-C* oscillator constitutes a good subject for an experiment dealing with the equation for simple harmonic motion. Theoretical expressions obtained for the frequency and for the damping coefficient can be verified by experiment, and either positive or negative damping may be had. With commonly available apparatus, the period may be made several seconds long. The oscillations and the growth or decay of amplitude with time can thus be observed directly with a milliammeter biased to read zero at its center. Familiarity with a simple two-stage triode amplifier constitutes the needed background in electronics for this experiment.

The *R-C* oscillator is unusual in that it can generate a sinusoidal current without using any inductance to form a resonant circuit. The theory of such oscillators has been adequately treated[1,2] by alternating circuit theory with complex quantities. The analysis given below shows how the same circuit can be analyzed without using complex quantities, thereby making the *R-C* oscillator a suitable subject for a laboratory experiment at a more elementary level. As such, it has a number of desirable features that will appear in what follows.

To analyze the *R-C* oscillator without using complex quantities, consider the circuit shown in Fig. 1. The circuit contains an amplifier as indicated, with an input voltage E_i and an output voltage E_0. Assuming that E_0 is proportional to E_i, we may write

$$E_0=AE_i, \tag{1}$$

where A is the voltage amplification. Let us also assume that the input current to the amplifier is negligible compared to the output current. Under these conditions, the relationships that hold between the instantaneous currents and voltages will be

$$I_s=I_c+I, \tag{2}$$

$$E_0=E_i-(1/C_1)\int I_s dt-I_sR_1, \tag{3}$$

$$E_i=-(1/C)\int I_c dt, \tag{4}$$

$$IR=(1/C)\int I_c dt. \tag{5}$$

[1] H. H. Scott, Proc. Inst. Radio Engrs. **26**, 226–235 (1938).
[2] Terman, Buss, Hewlett, and Cahill, Proc. Inst. Radio Engrs. **27**, 649–655 (1939).

Combining the above equations to eliminate all voltages and currents except I gives

$$d^2I/dt^2+2\alpha(dI/dt)+\omega_0^2I=0, \tag{6}$$

where

$$2\alpha=(1/CR)+(1/CR_1)+(1/C_1R_1)-(A/CR_1) \tag{7}$$

and

$$\omega_0=1/\sqrt{(C_1R_1CR)}. \tag{8}$$

Equation (6) is the well-known equation for simple harmonic motion with damping. Its solution may be written in the form

$$I=I_0e^{-\alpha t}\sin(\omega t+a), \tag{9}$$

where

$$\omega^2=\omega_0^2-\alpha^2. \tag{10}$$

Thus the current I will be a sinusoidal current, and the amplitude will increase, decrease, or remain constant depending on whether α is negative, positive, or zero, respectively. According to Eq. (7), α will be zero when

$$A=(R_1/R)+(C/C_1)+1. \tag{11}$$

It follows that the amplitude of the oscillations will increase if the amplification is greater than this critical value, and decrease if it is smaller.

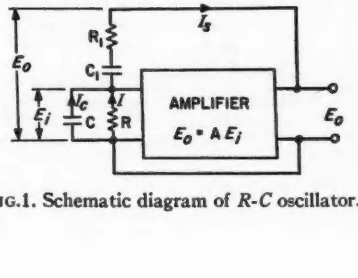

Fig. 1. Schematic diagram of *R-C* oscillator.

If R is made equal to R_1, and C equal to C_1, the critical value of A for constant amplitude is three. For constant amplitude, ω is equal to ω_0, and the angular frequency of the oscillations will be given by Eq. (8). The expressions for the frequency of the oscillations and the critical value of the amplification as given in Eqs. (8) and (11), respectively, are the same as those resulting from the analysis using complex quantities.

The R-C oscillator is a good subject for a laboratory experiment dealing with the equation for simple harmonic motion. The observed frequency can be checked by calculations involving known capacitance and resistances. Both positive and negative damping can be obtained in practice, and the observed condition for constant amplitude can be checked by calculations. It is easy to obtain slow oscillations so that the eye can follow both the oscillations and the changes in the amplitude of the oscillations. For example, if $C_1=C=1$ μf, and $R_1=R=0.27$ meg, the oscillations will have a period of about 1.7 sec. At this frequency, the pointer of a small pivoted-coil meter placed in the circuit will follow the oscillations of the alternating current. Higher frequencies to be observed with an oscillograph can also be obtained with the same circuit by simply changing the resistances and capacitances.

The complete circuit for an R-C oscillator can be made very simple as indicated in Fig. 2. The data given there apply for a twin-triode tube 6SN7, with both triodes shown in the figure in the same tube envelope. The use of grid biasing batteries rather than cathode resistors makes it relatively easy to compute amplification for an

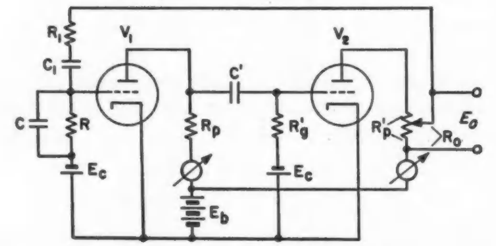

FIG. 2. Circuit of an R-C oscillator using 6SN7 twin triode. $R=R_1=0.27$ meg; $C=C_1=1$ μf; $C'=10$ μf; $R_g'=1$ meg; $R_p=1000$ ohms; $R_p'=1000$ ohms; $E_c=-4.5$ volts; $E_b=230$ volts.

experimental verification of Eq. (11). With the circuit shown, the value of A in Eq. (11) will be given by the equation

$$A=[\mu R_p/(r_p+R_p)]^2 R_0/R_p', \tag{12}$$

where μ is the amplification factor of either triode and r_p is the plate resistance. The factor in the brackets is the voltage amplification of one triode stage, and the amplification of the two similar stages will be approximately the square of that for one stage, when coupled as indicated. For operating conditions specified, μ is approximately twenty, r_p is 7500 ohms, and this gives

$$A=5R_0/R_p'. \tag{13}$$

Thus by sliding the potentiometer tap to change R_0 from zero to R_p', A can be varied from zero to five including the critical value of three required for oscillations of constant amplitude. As long as the amplitude of oscillations stays within the linear range of the amplifier, the plate current of either tube will be proportional to the current I of Eq. (9). The 6SN7 tube can be biased so that a 10-mil meter placed in either plate circuit will indicate positive and negative currents on either side of a quiescent value.

Commercial circuits for R-C oscillators commonly use a ballast tube as a bias resistor in the cathode lead of the first tube. This, along with other modifications of the circuit, serves to stabilize the oscillator by automatically decreasing the amplification as the amplitude of the oscillations increase. This is not needed in a laboratory experiment where A is to be varied manually, and the circuit shown is inherently stable enough so that oscillations of constant amplitude can be obtained by manual adjustment of R_0.

A simple R-C oscillator as described here can also be used as an ac generator for demonstrations that require a very low frequency. For example, the phase relationships that exist in connected ac paths can be directly demonstrated using small meters with their zero at the center of the scale. The oscillations may be made slow enough so that the eye can follow the pointers as they indicate approximately instantaneous values.

Lines of Force in Electric and Magnetic Fields

Joseph Slepian
Research Laboratories, Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania
(Received June 12, 1950)

Assemblages of lines of force are widely used to represent electric and magnetic fields, the density of lines in the neighborhood of a point being proportional to the intensity of the vector at that point. It is not generally recognized, however, that the properties of continuity and individuality of such lines are irrelevant so far as observable electromagnetic phenomenon are concerned.

Contrary to widely held views, it is not possible generally to represent the magnetic field of a linear circuit carrying a steady current by such an assemblage of lines of force which are everywhere continuous. This is because the continuous magnetic lines of force do not generally form simple closed curves, which each link the circuit just once. The interlinkage which is significant is that between the circuit and arbitrarily chosen closed paths of integration of the magnetic vector.

THE electric and magnetic fields appear in Maxwell's equations as vector fields, that is, as vector functions of position in space which are also functions of time. Such a vector function assigns to each point in space at any moment in time, a magnitude and a direction, and that is all. Maxwell's theory is a successful "local action" theory in that a knowledge of these vector fields within a given local region is sufficient to account for or describe adequately all observable macroscopic electromagnetic phenomena within that region. If the region is surrounded by a closed geometric surface, then a knowledge of the field vectors at all points of the surface (and for all time), is sufficient for determining all "outside" influence upon the electromagnetic phenomena within the surface. Further detailed information as to the fields or electromagnetic goings-on outside the surface is unnecessary.

A proper geometric representation of a vector field would be the obvious one of placing an arrow at each point in space with the appropriate direction and length. Such a picture assigns to each space-point a direction and a magnitude and nothing more, and therefore introduces no irrelevant elements. However, another geometric picture is widely used, which, while it does generally properly assign the appropriate direction and magnitude to each point of space, nevertheless may induce the less critical user to introduce irrelevant elements which correspond to no observable electromagnetic phenomena. This is the method of using assemblages of continuous curves, lines of force, spaced so as to give a density at each point of space equal to the magnitude, there, of the vector being represented.

Failure to recognize the irrelevance of certain features of this picture leads many students to incorrect expectations as to the nature of the electric and magnetic fields. Particularly among electrical engineers, Faraday's law of induction is frequently presented in terms of this picture with an associated incorrect portrayal of the magnetic field. Meaningless, or at least not uniquely meaningful terms, such as common and leakage flux of two circuits, are found widely in engineering literature. It is the purpose of this paper to point out the irrelevant elements of the lines of force representation of electric and magnetic fields and some of the incorrect expectations aroused by these irrelevant elements.[1]

Line of Force Representation of a Vector Field

A line of force of an electric or magnetic field is usually defined as a directed continuous curve which at each of its points has the direction of the electric or magnetic vector there. At any point where the vector magnitude is not zero, we may uniquely draw such a line of force. An assemblage of such lines of force is a satisfyingly vivid means

[1] More detailed discussion of these points will be found in the Electrical Essays of the author, Elec. Eng. **66**, 872 (1947); **67**, 58 (1948); **67**, 530 (1948); **67**, 564 (1948); **68**, 449 (1949); **68**, 519 (1949); **68**, 518 (1949); **68**, 615 (1949); **68**, 613 (1949); **68**, 678 (1949); **68**, 677 (1949); **68**, 763–4 (1949); **68**, 762 (1949); **68**, 878 (1949); **68**, 984–5 (1949); **68**, 1081–2 (1949).

for portraying the directions of the vectors of the vector field at the various points of space.

In the immediate neighborhood of a point where the vector intensity is not zero, we may choose the density of lines of force in the assemblage as proportional to the vector magnitude. Then if the divergence of the vector field is zero there, we may prolong the same continuous lines of force away from the region, and the assemblage of these prolonged lines of force will continue by its density to represent the magnitude of the vector field, the lines of force spreading where the vector field weakens, and converging where the vector field strengthens.

If the divergence of the vector field is not zero, then the density of the prolonged lines of force will not continue to correspond to the vector intensity. We may, however, now keep up this correspondence by ending some of the lines or starting new lines as the need arises. Then the density of lines stays proportional to the vector intensity, but lines end or new lines begin with a volume density depending on the sign, and proportional to the magnitude, of the divergence of the vector field. Throughout the region where this assemblage of lines of force can be so constructed, we get a vivid portrayal of the intensity as well as the direction of the vector field with the line endings and beginnings portraying the divergence.

The construction of the assemblage of lines of force as described above may be stopped in two ways. We may arrive at a point where the field intensity is zero. Or the lines of force may sweep around and return to a region where we have already selected and drawn lines of force. It is the latter case which we shall discuss in this paper.

Irrelevance of Individuality and Continuity of a Line of Force

Since it is only the density and direction of the lines of force of the assemblage which are related to the electric vector or magnetic field, it should be quite evident that the individuality and continuity of individual lines of force of the assemblage are entirely irrelevant properties, so far as any observable electromagnetic phenomenon is concerned, postulating as we do that the vector fields are entirely sufficient for describing such phenomena. As we prolong the lines of force, we may at any point end all the lines and start new ones with the same density, without in any way impairing the correspondence between the assemblage of lines of force and the vector field. The assemblage of broken lines serves just as well as the assemblage of continuous lines of force.

No observable electromagnetic phenomenon can exist which involves two points in space, and which depends upon there being a *continuous* line of force joining the points. Such a phenomenon would contradict our postulate of the complete sufficiency of the local *vector* fields for describing local phenomena.

Magnetic Lines of Force Are Generally Not Closed Curves

The complete permissibility of arbitrary breaks in the continuity and individuality of the lines of force in an assemblage representing a vector field comes as a shock to many people, particularly for the case where the divergence of the vector field is zero. It is still more of a shock when they learn that the complete representation of such a divergence-free field by an assemblage of lines of force without such arbitrary discontinuities is, in general, not possible.

Consider the steady magnetic field produced by a current-carrying loop of wire, where the loop lies almost but not completely in one plane. We start an assemblage of lines of force from a limited region and follow them continuously around the wire. After circling the wire, the lines of force return to the region from which they started, but since the loop is not plane, the individual lines do not return to the individual identical points where they originated. The lines may not then be continued on into the initial region, since that would alter the density of lines there. The lines can only be stopped. The density of lines which are thus stopped in this region is just equal to the density of lines which were begun there, so the condition for zero divergence is satisfied. Thus we may construct a satisfactory lines of force representation of the magnetic field of the loop, but the lines of force cannot be completely continuous curves, but must be broken at least once on the average for

each circumscription of the wire, if they are to have the proper density.

The author has inquired among a large number of trained physicists and engineers. Much more than half of them believed that the lines of force of the magnetic field of a current loop necessarily were closed curves each linking the loop once. This belief seems to arise from the belief in the universal possibility of the representation of the vector field by an assemblage of *continuous individual* lines of force. Actually, however, the lines of magnetic force will form closed curves each linking the wire loop once, only if the loop lies wholly in a plane.

Interlinkage of Magnetic Lines and Electric Circuits

The interlinkage of two closed curves in space has an intuitive meaning to most of us. It is a topologically invariant property, in that no continuous deformation of the closed curves with neither cutting the other, can alter that mutual property. The interlinkage may be quantitatively defined as the algebraic total number of intersections of either curve with any simply connected two-sided surface bounded completely by the other curve. In this definition, an arbitrary direction along each curve is taken as positive. From this, by a right-hand rule, a positive side is selected for the bounded surface, and thence one arrives at an intersection being called positive if it is made by the intersecting curve passing through the surface from the negative side to the positive, and negative if it passes through the surface in the inverse direction. The interlinkage thus quantitatively defined has for value a positive or a negative integer, or zero.[2]

If one or both of the two curves is open, or not closed, then interlinkage loses its sharp geometrical meaning. It is quite evident that in this case continuous deformation of the curves can always completely undo any degree of apparent, intuitively conceived, interlinkage. The quantitative definition given above also fails, since the open curve no longer completely bounds a simply connected two-sided surface. Since, as has been indicated in the previous section, the lines of magnetic force in general are not closed continuous curves, it has no meaning to speak of the interlinkage of a line of magnetic force with a circuit.

It is quite evident from his writings that Faraday had positive views as the physical significance of continuous lines of force which were quite different from those of this author. It seems highly probable, however, that Faraday also believed that magnetic lines of force always formed closed curves, and this last is demonstrably not true. That Faraday's apparently closed magnetic lines linked electric circuits undoubtedly gave him satisfaction as it apparently gave the magnetic field an intimate and unbreakable association with electric current similar or parallel to the association of the electric field with the electric charge upon which its open electric lines of force terminate.

Maxwell attempted, and successfully, to reduce Faraday's ideas to mathematical form. But in so doing, in his equations at least, he was obliged to strip the irrelevancies from Faraday's ideas. Gone from the equations were the continuous lines of force, reaching from plus charge to minus charge, or forming closed curves linking circuits. There remained only the vital essence, the vector electric and magnetic fields, each giving at each point of space a magnitude and a direction and that is all.

However, Maxwell's equations still assert relations involving interlinkages between magnetic fields and current circuits, but these relations are somewhat subtler than those suggested by Faraday. Applying Stokes' theorem to the appropriate Maxwell's equation we deduce the following for the case of a single electric circuit, which is of course closed. Consider any closed curve in space. Then the integral of the vector magnetic field around that closed curve is proportional to the product of the current magnitude and the number of interlinkages with the electric circuit of the closed curve of integration. The interlinkages then are not of the circuit with closed magnetic lines of force, but of the circuit with arbitrary closed paths of integration.

Likewise, applying Stokes' theorem to that Maxwell's equation which expresses Faraday's law of induction, we find that the integral of the vector electric field around any closed curve is proportional to the time rate of change of the

[2] Alexandroff and Hopf, *Topologie* (Verlag. Julius Springer, Berlin, 1935), pp. 410-417.

integral over any simply connected two-sided surface bounded by that curve of the normal component of the vector magnetic flux density. In the special case that the magnetic flux density lines do all form closed curves, which they generally do not, then an assemblage of them can be constructed whose density is everywhere proportional to the magnetic flux density magnitude, and then the surface integral just referred to will be proportional to the sum of the interlinkages with the closed electric circuit of all the individual closed flux line curves of the assemblage. It is this property of the surface integral for a very special case which causes it to be known even in the general case as the "flux linkage of the circuit."

Maxwell's equations and their integrals discussed above are widely known and correctly used by physicists and engineers. However, also widely accepted among these physicists and engineers are the erroneous elements of Faraday's anticipatory ideas, namely, that there is physical significance in the continuity and individuality of electric and magnetic lines of force, and that the lines of force of the magnetic field form simple closed curves.

The Solution of Differential Equations by Electrical Analog Computers

Joseph L. Ryerson
Evansville College, Evansville, Indiana
(Received May 31, 1950)

Electrical systems for solving total differential equations are classified as high speed or low speed with citations of existing commercial equipment. A symbolism is proposed for the physical components which perform the mathematical operations without elaborating upon the technical details of their design.

The solutions of simultaneous linear algebraic equations, simultaneous differential equations, and differential equations with variable coefficients are discussed and illustrated. The article contains fourteen illustrations of symbols and system wiring schemes.

BECAUSE of the rapid growth of general purpose analog computing equipment, during recent years many engineers and physicists have been unaware of the vast problem-solving facilities now available. Many problems have never been formulated rigorously because of the man-hours required to obtain a solution, assuming a solution to be possible with existing mathematical knowledge. As an illustration, A. C. Hall[1] has pointed out that equations characterizing aircraft motions are commonly of order between twenty and thirty. The existing commercially available office-size analog computing equipment will now enable a scientist to obtain a numerical solution of practically any total differential equation or group of simultaneous equations to a precision of approximately 0.1 percent.

The purpose of this presentation is not to discuss the technical details involved in the construction of a particular computer but to point out the philosophy of an analog computing system. It is intended to give an over-all view of what is happening in the problem-solving process and arouse in the reader a creative interest in analog computers.

Broadly speaking, an analog computing system is one which substitutes a convenient physical system for an inconvenient physical system, the convenient physical system being described by differential equations which are numerically identical to those describing the inconvenient system. In other words, a convenient system is set up to simulate an inconvenient system. For example it might be required to study the motion of a locomotive described by the equation

$$F = Kv + M dv/dt,$$

where v and t denote velocity and time, respectively.

A convenient physical system which is mathematically identical would be an electric current i

[1] A. C. Hall, Elec. Eng. **69**, No. 5, 433–436 (1950).

flowing in an inductance coil as described by the following equation:

$$E = iR + L di/dt.$$

This circuit may conveniently be set up in a small space with the numerical values of E, R, and L identical, respectively, with F, K, and M. With the value of the initial conditions identical, a curve of i *vs* t will be numerically identical to a curve of v *vs* t. It may be said that in this particular instance electrical current is analog to velocity. The precision of the numerical solution obviously depends upon the precision of the measurement of the variables forming the analog system. For precisions requiring more than three or four significant figures, analog equipment becomes very costly. It may also be inferred from the above example that the physical volume of the analog equipment must increase with the complexity of the problem.

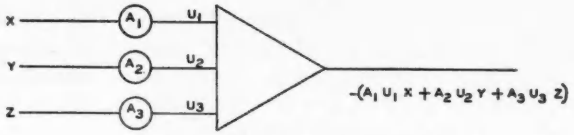

FIG. 1a. Symbol for summing amplifier.

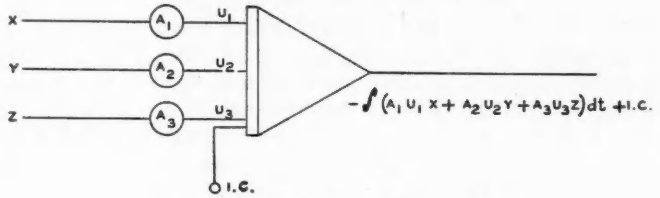

FIG. 1b. Symbol for integrating amplifier.

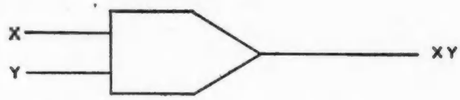

FIG. 1c. Symbol for multiplying device.

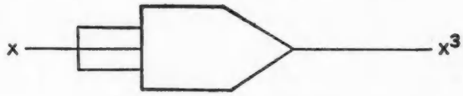

FIG. 1d. Symbol for cubing device.

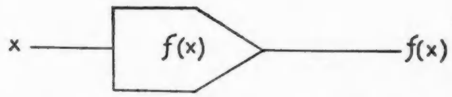

FIG. 1e. Symbol for function generating device.

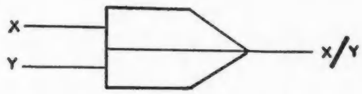

FIG. 1f. Symbol for dividing device.

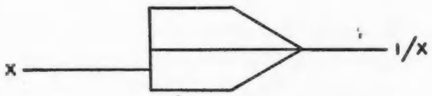

FIG. 1g. Symbol for reciprocal-taking device.

It must not be concluded from the above example that analog equipment is necessarily electrical. The problem could just as well have been solved by measuring the angular velocity of a shaft attached to a flywheel, viscously damped under constant torque. Nevertheless, the writer must concede that electrical circuits provide the most versatile and convenient analog systems. It is quite simple to measure voltage to any required precision; consequently, existing commercial computers use voltages for the variables in electrical analog systems.

From the examples cited thus far the reader probably has the impression that in order to solve the differential equation describing a physical system by means of an electrical analog computer, he must deliberately set up an electrical circuit having a differential equation identical in form to the system under observation. This is entirely unnecessary, as the analog system may be segregated into separate components which perform physical operations on voltages fed into the components. The components of the system then become physical symbols which the operator may synthesize, by means of convenient electrical connectors, into the equation or system of equations he wishes to study. The operator need only know the symbolism of the components and does not concern himself with the interior physics of the equipment.

The operations on voltages that must be performed by the equipment are the usual opera-

tions that must be performed in the solution of problems, namely, addition, subtraction, multiplication, division, differentiation, integration, involution, and evolution. The process of addition and subtraction is usually accomplished by means of a summing amplifier.[2] This amplifier has several input terminals and one output terminal, all voltages being measured with respect to ground. The input voltages may be fractionated by means of a potentiometer and then multiplied by an amplification constant. The output of the amplifier is the negative of the sum of all the input voltages multiplied by their respective constants. The symbol[3] for this amplifier is shown in Fig. 1(a). If a feedback capacitor is connected between the output and input of the amplifier, it will integrate the sum of the input voltages with time. The symbol for this amplifier is reproduced in Fig. 1(b).

Hereafter, in the case of summing and integrating amplifiers the complete value of the coefficient of the variable will be inserted within the circle of the potentiometer.

The multiplication of variable voltages presents a formidable technical problem that can, however, be solved by the use of instrument servos,[4] nonlinear impedances, or variable mu tubes. The author will not consider the technical details but simply designate a symbol for multiplication, as follows [Fig. 1(c)]. Where it is required to raise a number to a power, two or more input terminals will be tied together as indicated in Fig. 1(d). A special function will be designated by the symbol in Fig. 1(e), where the name of the function of X is written within the diagram.

The symbol for division will be selected as in Fig. 1(f). Wherever a reciprocal is required, the upper or numerator terminal will not be shown. Fig. 1(g) represents this case. Note that all symbols have a directional arrow head on the right-hand side. This indicates that the device is an unilateral operator. In other words, the inverse operation will not be performed by applying a voltage to the output terminal.

The remainder of the paper will be devoted to the synthesis of equations by appropriate circuit arrangements of component operators. As a first example a group of linear simultaneous equations will be solved.

$$A_{11}X+A_{12}Y+A_{13}Z+K_1=0, \tag{1}$$

$$A_{21}X+A_{22}Y+A_{23}Z+K_2=0, \tag{2}$$

$$A_{31}X+A_{32}Y+A_{33}Z+K_3=0. \tag{3}$$

The first step is to divide out the coefficient of X in Eq. (1), Y in Eq. (2), and Z in Eq. (3). The equations then become

$$\begin{aligned}X+A_{12}'Y+A_{13}'Z+K_1'&=0,\\ A_{21}'X+Y+A_{23}'Z+K_2'&=0,\\ A_{31}'X+A_{32}'Y+Z+K_3'&=0.\end{aligned}$$

The summing amplifiers are then set up to solve each equation for a different variable as indicated in Fig. 2. The arrows indicate that the particular variable is carried to a measuring station where its value is recorded as a solution. Note that one amplifier is required for each equation.

It would have been possible to solve each equation for the same variable and then by means of external supply voltages adjust the values of the remaining variables until all similar variables become equal. The magnitudes of the variables at the condition of equality would be the solution of the system. Such a trial and error procedure would eventually lead to a solution but may be accomplished automatically by driving the amplifiers with their own outputs as has been done, thus eliminating the need for an external source of voltage. At first glance, such a procedure may seem to violate the law of the conservation of energy; but it should be remembered that each amplifier contains a source of energy.

The circuit of Fig. 2 is greatly simplified for purposes of clarity and continuity of thought. In reality, the solution of systems of linear algebraic equations is more difficult than the solution of differential equations because of stability difficulties. The circuit of Fig. 2 demands an instantaneous balance which is beyond the capabilities of any physical system and usually results in a random oscillatory change in the output voltages representing the desired solution. Such

[2] *Electronic Instruments*, M.I.T. Radiation Series (McGraw-Hill Book Company, Inc., New York, 1949), Vol. 21.
[3] This summing amplifier, integrating amplifier, and potentiometer symbolism has been adopted by the Reeves Instrument Corporation of New York.
[4] Reeves Instrument Corporation uses servosystems.

a condition is overcome in practice by modifying[5] the original system of linear equations by the addition of first-order derivatives of the unknown quantities, thus converting to a system of differential equations which do not present stability problems. The first-order derivatives must be selected by a mathematical procedure which will cause the inherent errors which occur at the outputs of the summing amplifiers to be exponentially attenuated in the solution.

As a second example, a system of simultaneous differential equations having constant coefficients will be solved. The operator P is used to indicate differentiation with respect to time.

$$\begin{aligned}
&P^2X+B_{12}P^2Y+C_{12}P^2Z+A_{11}PX+B_{11}PY \\
&\qquad +C_{11}PZ+A_{10}X+B_{10}Y+C_{10}Z=0, \\
&A_{22}P^2X+P^2Y+C_{22}P^2Z+A_{21}PX+B_{21}PY \\
&\qquad +C_{21}PZ+A_{20}X+B_{20}Y+C_{20}Z=0, \\
&A_{32}P^2X+B_{32}P^2Y+P^2Z+A_{31}PX+B_{31}PY \\
&\qquad +C_{31}PZ+A_{30}X+B_{30}Y+C_{30}Z=0.
\end{aligned}$$

The equations are solved by setting up three loops corresponding to the three variables. The first loop is set up to synthesize the quantities containing X in the first equation. These quantities are obtained by a succession of operations performed first on the term of highest order containing X. Identical procedures are followed for the second and third equations. The complete equations are then synthesized by feeding in data from the Y and Z loops to the X loop, from the X and Y loop to the Z loop, etc. The circuit which synthesizes the system of equations written above appears in Fig. 3.

Note the arrows on the connectors which are at the X, Y, and Z potentials. They indicate that these potentials are fed to an output device which consists of a recording voltmeter. At zero time all amplifier inputs are disconnected. When the solution is desired all amplifiers are connected simultaneously, all initial conditions having been previously adjusted. The solution itself is a curve of voltage *vs* time for the desired variables, the time interval extending from zero to infinity. The word infinity is used here in a limited technical sense and indicates the time required for the variable to reach asymptotic

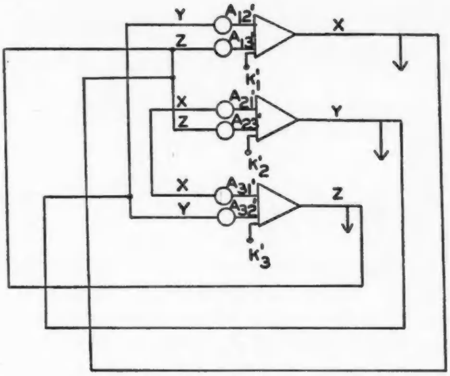

FIG. 2. Circuit for solution of linear simultaneous equations.

values or to indicate a definite approach to an asymptote. This time, of course, will be a relatively short finite period. If the computer is constructed to yield solutions to a precision of 0.1 percent, then the approach of a variable to within less than 0.1 percent of an asymptote will have no significance and the solution need not be continued farther.

A distinction should be made at this time between high and low speed computers. A low speed computer may complete a solution in from three seconds to five minutes, a high speed[6] computer may present a solution in five milliseconds. In the high speed device, the initial conditions are impressed at a high repetition rate, say 250 cycles per second, and the solutions appear at the same rate. The successive solutions are pre-

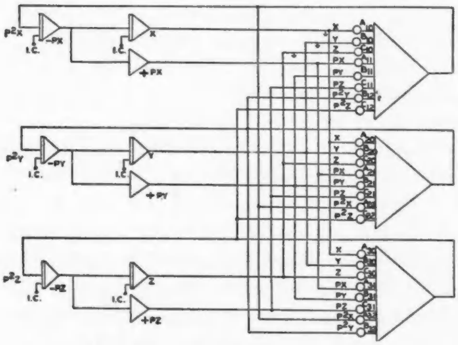

FIG. 3. Circuit for solution of linear simultaneous differential equations.

[5] Forrest L. Gephart, "Treatment of linear algebraic systems on an electronic differential analyzer," Applied Mathematics Group, Office of Air Research, Dayton, Ohio.

[6] High speed equipment is produced by the G. A. Philbrick Researches, Inc., of Boston, Massachusetts.

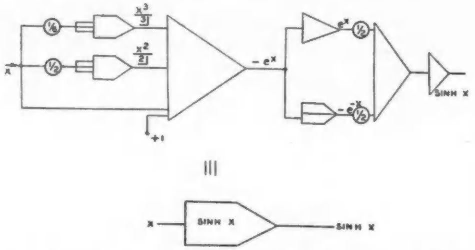

FIG. 4. Circuit for generation of sinh of input voltage X.

sented on the screen of a cathode-ray oscilloscope rather than as an ordinate of a recording voltmeter. It is usually necessary to scale down the time dimension so that infinite time elapses within the time base of the impressed initial-condition square wave. The scaling down of the time dimension is accomplished in the constant coefficients of the original differential equation. Each time variable is simply multiplied by the desired scale factor. It is the opinion of the author that low speed computers are inherently more precise than high speed devices because of the frequency-response difficulties encountered at high frequencies. In addition, the low speed equipment can use standard high precision direct current instrument movements as output devices. With the use of proper standards it may be possible to build a low speed computer capable of yielding solutions having more than four significant figures.

The differential equations which have been discussed thus far contained constant coefficients only. It is possible by means of the multiplying devices discussed earlier to solve equations containing variable coefficients. These coefficients may be either known or empirical functions. Many known functions may be approximated to within the required precision of the instrument by use of the first three or four terms of the infinite series describing the function. As this procedure is somewhat cumbersome and requires a large volume of equipment, it may be more expedient to wind a special potentiometer having an output voltage which is the desired function of the angle of rotation of the contact arm. This angle of rotation may be caused to vary linearly with an impressed voltage by means of a servomechanism so that the output voltage of the special potentiometer is ultimately the desired function of the impressed voltage. Systems using servomechanisms are adapted for use only in slow speed computers and their use is justified mainly by the precision that may be built into such function generators. Occasionally it may be too costly to build a special potentiometer to generate a function. In this case it is possible to make a paper and ink graph of the function, which is inserted into a specially prepared device in which the "X" position of a cross hair is determined by a servomechanism to which is being fed the independent variable from the computing circuit. The cross hair is manually positioned over the curve by an operator who turns a crank attached to the contact arm of a linear potentiometer. The output of this linear potentiometer is then directly proportional to the "Y" of the graph. This "Y" voltage, which is the desired function, is fed back into the computing circuit at the appropriate point. The operator may, of course, be replaced by an elaborate photoelectric following device. The high speed computers cannot rely on the inherently low speed procedures described above but must develop special nonlinear circuit[7] elements which must have stable calibration curves. It is also possible, in high speed circuits, to develop shaping circuits consisting of passive linear elements in three terminal networks. These networks have the property of acting on input square waves so as to yield output wave shapes that are a predetermined function of time.

In Fig. 4 is shown a circuit for the generation of $\sinh X$. This circuit is based on the infinite series method for the approximation of e^X using the circuit symbol discussed earlier. While it is possible to convert $\sinh X$ directly into series form, the writer has deliberately chosen a more elaborate circuit for purposes of illustration. Symbolically, the entire circuit may be replaced by the single function generator to which it is equivalent.

The generation of time functions does not require the elaborate multiplying equipment described heretofore but can use the more or less standard integrating amplifiers. Shown in Fig. 5 is a circuit which may be used to generate the function $\sinh bt$. This function will not be gen-

[7] High speed computers using these methods were developed during World War II. See reference 2.

erated by series approximation but by means of the known solutions of simple differential equations. These solutions are added to form the desired time function. The function $\frac{1}{2}e^{\pm bt}$ may be obtained by solution of the equation $0=PX \pm bX$. The initial conditions must be $t=0$, and $X=\frac{1}{2}$. The two solutions for positive and negative values of b may be added in a summing amplifier to obtain the desired function $\sinh bt$. Note that the equivalent symbolic time function generator shown in Fig. 3 has no input terminal. This is equivalent to stating that time need not be fed into the circuit in order to be operated upon but already exists as part of the physics of the circuit. The voltage that is actually fed into the circuit is, of course, the initial condition voltage.

Because of the frequent occurrence of trigonometric functions, it is justifiable to construct a special device for the generation of these functions. Probably the most widely used device for this purpose is the electromagnetic vector resolver.[8] This unit is essentially a two-phase wound rotor induction motor. Two independent windings are provided on the stator which produce magnetic fields in space quadrature, and two independent windings are provided on the rotor which produce magnetic fields in space quadrature. Within the linear portion of the iron the field magnitudes produced by these windings are in direct proportion to the voltages applied to them. The resolver is used with alternating voltages of a single frequency which are in time phase, and one end of all four windings may be connected to a common ground. If voltage is applied to one of the stator coils and one of the rotor coils is oriented within angle θ of it, then the voltage appearing on the rotor coil will be $\cos\theta$ times the stator coil voltage. Since the other rotor coil is at right angles to the coil considered, the voltage induced in it will be the stator voltage times the $\sin\theta$. The stator voltage is then resolved into two components at right angles to each other. If the voltage applied to the first stator coil is V_{S1} and the voltages induced in the rotor coils are V_{R1} and V_{R2}, then the relation-

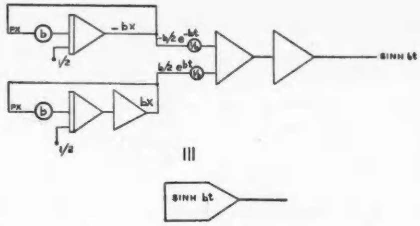

FIG. 5. Circuit for generation of sinh of time function bt.

ships described above may be written as follows:

$$V_{R1}=V_{S1}\cos\theta,$$
$$V_{R2}=V_{S1}\sin\theta.$$

If a voltage V_{S2} is applied to the remaining stator winding, additional voltages are induced in the rotor windings, which are the stator voltage times the sine and cosine of the angle increased by 90°. The complete rotor voltages become

$$V_{R1}=V_{S1}\cos\theta-V_{S2}\sin\theta,$$
$$V_{R2}=V_{S1}\sin\theta+V_{S2}\cos\theta.$$

If a servomechanism is used to drive the rotor of the resolver and a fixed stator voltage of 100 volts, for example, is maintained on one of the stator windings, it can then be used as a trigonometric function generator. Wherever other than sine or cosine functions must be generated, a dividing circuit must be used in conjunction with the resolver. When the resolver is used to obtain trigonometric functions, no special circuit symbolism will be used other than the symbol for a function generator already developed. It should be borne in mind that the resolver is an ac device which will, therefore, require special modulators when used in conjunction with dc computing circuits. It may be simpler to use sinusoidal potentiometers in many cases where simple trigonometric functions are required.

The electromagnetic vector resolver finds its greatest utility in the transformation of coordinates. It will transform Cartesian to polar coordinates or polar to Cartesian with equal facility. If it is desired to transform from Cartesian to polar coordinates, the two voltages X and Y are fed in on the stator coils. The resulting magnetic field in the interior of the resolver will be proportional to $(X^2+Y^2)^{\frac{1}{2}}$, which is, of course, the radius vector $\mathbf{R}$ of the polar system. If one

[8] *Components Handbook*, Radiation Series (McGraw-Hill Book Company, Inc., New York, 1949), Vol. 17, pp. 340–345.

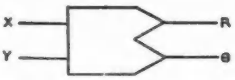

FIG. 6a. Symbol for transformation of Cartesian coordinates to polar coordinates.

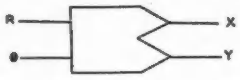

FIG. 6b. Symbol for transformation of polar coordinates to Cartesian coordinates.

of the rotor windings is used as a source-of-error signal which is fed into a servosystem which is driving the rotor of the resolver, the remaining rotor coil will take up a position such that the full voltage $(X^2+Y^2)^{\frac{1}{2}}$ will appear at its terminal. The angular displacement of the rotor will be $\tan^{-1}XY$, which is the remaining coordinate of the polar system. The angular position of the rotor is converted into a voltage by means of a linear potentiometer which has its contact arm attached to the shaft of the rotor. In order to convert polar coordinates into Cartesian, the voltage representing the radius vector is fed into a single stator winding, and the voltages appearing on the rotor windings are the corresponding Cartesian coordinates X and Y, where $X=R\cos\theta$, and $Y=R\sin\theta$. The angle of orientation θ of the rotor coils with respect to the stator is fed directly to the shaft from a servomechanism which has a rotation directly related to the applied voltage, which in this case represents θ, the angular coordinate of the polar system. The transformation of coordinates does not actually require special equipment as discussed above, but may be performed with any equipment capable of performing the indicated mathematical operations. The resolver is merely a compact device for performing a frequently desired transformation. As the specific circuits for performing coordinate transformations may vary, a single circuit symbol will be introduced at this point which may be used to represent uniformly any indicated coordinate transformation (see Fig. 6).

As a final illustration, a somewhat generalized third-order equation having variable coefficients will be solved. The procedure is first to perform a succession of integrations on the derivative of highest order. The end product of this series of integrations is the variable sought. The variable is then fed into the individual function generators which represent the variable coefficients. These functions are multiplied by the desired time func-

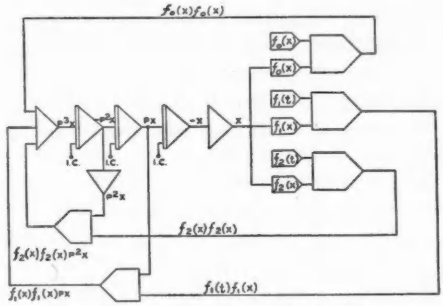

FIG. 7. Circuit for the solution of differential equations with variable coefficients.

tions and then by the desired derivatives. All of the resulting terms are then added in a summing amplifier, the output of which is the highest order derivative. The circuit representing the solution of the following equation is shown in Fig. 7. The symbol $f(t)$ or $f(x)$ denotes any arbitrary real function of t or x:

$$P^3X+f_2(t)f_2(x)P^2X+f_1(t)f_1(x)Px+f_0(t)f_0(x)=0.$$

It should be borne in mind that each function generator itself could be a rather elaborate circuit. The solution will be the voltage X plotted as a function of time.

In closing, a word of caution should be injected regarding the limitations of electrical analog computing equipment. Since these devices depend for their indications upon the transmission of voltages through circuit elements, the physical limitations and nonlinearity of these elements should be continually borne in mind. Suitable means for indicating over-voltages at every crucial point should be provided, since it it practically impossible to anticipate voltages without a knowledge of the solution of the problem. Obviously, the solution will be distorted by any saturation effects within the computing circuit.

It is hoped that the compact symbolisms employed will not lead the reader into an erroneous belief that all of the technical problems associated with computer design have been solved. Much is yet to be desired, such as simple, precise multiplying and dividing units. A versatile general purpose function generator would be of great value. The problems of analog computer design are the problems of precise physical measuring instruments and rightfully belong in the

sphere of the physicist rather than that of the engineer or mathematician. It is hoped that this article will inspire many physicists, heretofore unaware of analog computer possibilities, to creative efforts directed toward expansion of the computer art.

As a starting point for those interested in the interior physics of the operational components of electrical analog computers the following carefully selected references are cited in addition to those included in the footnotes: Francis J. Murray, *The Theory of Mathematical Machines* (King's Crown Press, Columbia University, New York, 1947); Ragazzini, Randall, and Russell, "Analysis of Problems in Dynamics by Electronic Circuits," Proc. Inst. Radio Engrs. **35**, 444 (1947): McCann, Wilts, and Locanthi, "Electronic Techniques Applied to Analogue Methods of Computation," Proc. Inst. Radio Engrs. **37**, 954 1949). These references, especially the second and third, contain sufficient technical detail to enable one to begin construction of units.

Specialized Physics

C. Harrison Dwight
University of Cincinati, Cincinnati, Ohio
(Received April 23, 1950)

Physics courses offered in American universities may be classified as either "Standard" or "Specialized." The former stress all parts of physics more or less equally, and laboratory work accompanies the lectures. The need for the specialized type of course is apparent to students in fields where a knowledge of the subject is very important in their training but where they do not need, and usually do not have time for, all of the topics covered in the standard course. Such students are those in Architecture, Business Administration, Music, Applied Arts, and so forth. The author briefly analyzes the two types of courses, emphasizing the advantages of each, and calls attention to the fact that both types of courses are offered at the University of Cincinnati.

THERE are two schools of thought among teachers of physics with regard to the problem of specialized courses, and it is likely that the same situation exists in other science curricula. On the one hand, there is the opinion that there is just one subject called "physics" and that there could not possibly be a physics for a musician, for an architect, and for a doctor. On the other hand, because of the wide variety of professions and individual interests, many educators realize that an introductory physics course should be provided which will fit most nearly the needs of the students as they go out from the classrooms and begin to apply the principles in their professions.

This somewhat controversial problem may be analyzed as follows. In the first place, a physical truth like Ohm's law, evolved from experiment and theory, is unchanged as the same teacher presents it to liberal arts students, or to engineers or to architects. It is the same even if some other teacher presents it! Newton's law of universal gravitation is not a direct fifth-power relationship in a physics course for architects and an inverse square relationship for majors in mathematics. We all agree that there *is* but one physics. The proponents of the standard course for all types of students maintain that the latter should study all of the subject that can be mastered in the allotted number of student-contact hours. The order of development of the general standard course in physics is more or less stereotyped. After a week of indoctrination into the language of the science at the start of the course, the student is introduced to mechanics, light, heat, and the other branches of the subject. The level of difficulty of the course is somewhat influenced by the average mathematical training of the class—never to be overestimated! The purposes of the traditional general course are (i) to acquaint the student with the scientific method of investigating nature, (ii) to instill logical mental habits, (iii) to familiarize the student with physical laws, (iv) to indicate the

everyday illustrations of these laws and (v) to develop a broad cultural background in science, which will consider the lives and works of great physicists, their relationships to the times in which they lived, and the common problems of the border sciences. A student who takes a full year of general physics, as just described, together with a laboratory course which has kept more or less in phase with it, emerges from his ordeal moderately qualified to go into more advanced courses in science, or into industrial research or into teaching at the preparatory school level. He will be able to understand something of the theory of the atom, which in these days has been elevated to the headlines; he will appreciate the many able articles on science which appear from time to time in the public press. This student has received an education in physics rather than a mere training in techniques.

Proponents of the specialized physics course do not regard the material covered in the standard curriculum as useless for their students—in fact, most of these teachers sigh for more time in which both to present everything they feel *should* be in their program and to emphasize especially topics germane to their groups of students. They would prefer to give their classes the entire gamut of instruction from weights and measures at the beginning to nuclear fission as the grand finale of radioactivity at the end of the school year. But the claims of technical courses on the working time of the preprofessional student have made it advisable for many colleges and universities to introduce abridged physics courses which will offer the student a little of each topic covered in the standard course but with a preponderance of attention and effort given to those items that he will later find most useful and profitable. Thus purposes (iii) and (iv) mentioned above are brought to the fore and the others are slighted deliberately by the instructor.

College catalogs list such courses as *Psychology of Advertising* and the *Chemistry of Foodstuffs*. Why shouldn't there be a *Physics of Music?* In a reputable institution each of these courses is taught by a man who is professionally competent to deal with the *entire* field, but who is at the same time sufficiently conversant with neighboring fields and is possessed of enough imagination to pick out salient applications so that he can make the specializing student feel that he is learning something worthwhile. It would seem to be an unnecessary demand upon a major in music to expect him to study much about light, electricity, and magnetism, but the serious student in music *can* and *should* be well versed in the concepts of force, energy, simple harmonic motion, and the general theory of waves; he should be familiar with the laws of vibrating strings, plates, and air columns, as well as with the physics of the orchestral instruments based upon these laws; he should know something of the subjects of audition and speech and how music is "put on the air." Finally, the student of music should know the main facts about the acoustical treatment of buildings. In the field of applied arts there is a considerable body of nonmathematical material of great importance to the student in certain parts of physics, such as light, color, tensile strength, the physical properties of substances, the effects of heat observed in ceramics, and the electrolytic restoration of ancient metallic art objects. The premedical student needs an entire course in general physics, together with special emphasis[1] upon x-ray techniques, electrotheraphy, the physics of the eye and ear, the optics of the microscope, the mechanics of muscle action, and the powerful and modern tools presented to medical research in the form of artificial radioactivity. For the student of business administration[2] a general survey of the entire field of physics is regarded as essential, but with a minimizing of the mathematical treatment and a building up of the industrial developments arising from researches in pure physics or in allied fields. How many physics teachers realize that Newton was a tolerably good business man—Master of the Mint of Great Britain?[3] Students of business administration want to know how physics can be applied to the problems of the private laboratory which may be a part of the industrial concern with which they are now, or will be, associated.[4]

[1] Otto Stuhlman, Jr. *An Introduction to Biophysics* (John Wiley and Sons, Inc., New York, 1943), first edition.

[2] C. H. Dwight, Am. Phys. Teacher 2, 111 (1936). (Lack of space now prevents instruction at present in laboratory work.)

[3] Sir John Craig, *Newton at the Mint* (Cambridge University Press, London, 1946).

[4] *Physics in Industry* (American Institute of Physics, 1937). (The Institute has also published a report on the teaching of physics to premedical students.)

What shall be said of the architecture student? Does he require a full course in physics? In most universities he is put into a standard engineering or liberal arts course but in a few instances he is given a specialized course which, while treating all portions of physics, puts paramount emphasis upon phases of great practical application in his field, such as statics, strength of materials, soundproofing, simple electrical circuits, heat transfer, moisture content, illumination and electrolytic action. If time permits, he should be presented with some information in meteorology and climatology which he may combine with his physics in the proper design of solar houses.

Three factors militate against the specialized course. The first is the reluctance of some chairmen of departments to admit the usefulness of such programs, the second is the dearth of instructors who could be spared from the general physics lecture and laboratory work, and the third factor is the relatively small number of students in many colleges who are taking any physics at all.

We may at this time ask the privilege of becoming somewhat personal. Both types of physics curriculum are offered at the University of Cincinnati—there are the standard programs for engineers and liberal arts classes (including both lecture and laboratory), and there are special courses for students in architecture, applied arts and business administration. The premedical students attend the same classes as those in liberal arts. Each special course has its own textbook and instructor, and all are offered on the "cooperative" basis.[5] The latter permits the meeting of a three-hour course, extending over three seven-week terms, actually but fifty times during the academic year, since there must be time out for registration, quizzes, and holidays. This means that the material covered must not only be specialized but severely abridged. We might describe in some detail the physics course offered, in the sophomore year, to the students of architecture in the College of Applied Arts. Calculus is taken concurrently, with statics and strength of materials following in the next year. As soon as it can be arranged, a three-hour laboratory period per week is to be given to architects, the experiments designed to familiarize the students with certain highly important aspects of their physics work as it applies to their field of interest.

The architectural physics course of fifty lectures embraces the following material. The letter L designates a topic for laboratory treatment.

1. Orientation in physics; units, etc.
2. Elements of linear motion
3. Force—concept and units (gravitational)
4. Systems of forces (force table, L)
5. Equilibrium of a rigid body (parallel forces, L); A truss
6. Nonlinear motion
7. Work and energy
8. Power and machines; tackles and jackscrews (friction, L)
9. Properties of matter (Young's modulus, L)
10. Hydrostatics and wind pressure
11. Pneumatics and the mechanics of gases; caisson work
12. Thermometry and temperature
13. Thermal expansion (L)
14. Measurement of heat; fuel values (L)
15. Phase change; heating systems (L)
16. Moisture; air conditioning (L)
17. Thermal conductivity (L)
18. Convection and radiation; radiant heating
19. Mechanical equivalent of heat (L); adiabatic processes
20. Meteorology and climatology; solar houses
21. Elements of wave motion; types of waves
22. Laws of vibrating strings and air columns; resonance (L)
23. Acoustics of an auditorium
24. Magnetostatics; location of buried magnetic objects
25. Electrostatics—hazards of dust, thunderstorms, etc.
26. Ohm's law and the simple dc circuit (L)
27. Resistance
28. Electrochemistry: The voltaic cell
29. Electrochemistry: Faraday's laws of electrolysis (L)
30. Joule's law; electrical heaters and lights (L)
31. Thermoelectricity and photoelectricity
32. Electromagnetism: motor law
33. The magnetic circuit
34. Electromagnetic induction: generator law
35. The ac generator
36. Inductance and capacitance in a series ac circuit
37. Transmission of electric power
38. Oscillating circuits: elements of radio communication
39. Electronics and x-rays
40. The electromagnetic spectrum
41. Reflection of light (L)
42. Refraction of light (plates, prisms, lenses, L)
43. Lens systems (L)
44. Optical instruments (eye, telescope, sextant, camera)
45. Principles of illumination; photometry (L)
46. Color and its applications (spectroscopy, L)
47. Interference
48. Diffraction

[5] See article by Clyde W. Park, J. Eng. Educ. **33**, 410 (1943) for a description of the inception of the cooperative system of engineering education.

49. Polarized light and its applications (L)
50. Photoelasticity and its applications

Each of these lectures is complete in itself, with references both to the textbook and to collateral readings. Numerous problems are assigned each day. Every effort is made to impress upon the student the importance of the physical principles in his chosen field of activity.

We believe that we are progressing along solid ground.

The Classical Motion of a Rigid Charged Body in a Magnetic Field

HERBERT GOLDSTEIN*
Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received June 1, 1950)

A frequently used classical analog to the quantum precession of a magnetic moment in a magnetic field is the motion in a uniform static magnetic field of a symmetrical rigid body charged so that the ratio of charge to mass density is uniform. In general, the figure axis of such a magnetic top both precesses and nutates, and the mechanism producing the nutation is qualitatively different from that responsible for the nutation of the familiar gravitational top. Unless the body is spherical the angular momentum vector also nutates in the course of the precession, a phenomenon caused by the deviation of the torque on the body from the customary, but here incorrect, expression $\mathbf{M}\times\mathbf{B}$. The uniform precession frequency of the magnetic top is *not* the Larmor frequency but involves in addition terms depending on higher powers of the magnetic field, a correction which corresponds to the quadratic Zeeman effect.

THE precession of a magnetic moment in a magnetic field is one of the most ubiquitous phenomena of modern physics. From the earliest investigations on the Zeeman effect through the latest nuclear resonance experiment it has played a vital role. In introducing the concept to the student, and even in much of our intuitive thinking about the Larmor precession, comparison is instinctively made to the familiar precession of a heavy top in a gravitational field. The exact classical analog, however, is not the heavy top but rather a charged rigid body in a magnetic field. Such a model shares with the nonclassical intrinsic "spin" of a fundamental particle the important characteristic that the magnetic moment is proportional to the angular momentum. The motion of a charged top differs both qualitatively and quantitatively from that of a heavy top, and affords considerable insight into the nature and genesis of the Larmor precession. A detailed discussion of the motion may therefore not be without interest.

To facilitate comparison of the heavy and charged tops Part I presents a summary of the salient features of the motion of the heavy top with one point fixed. This section also serves to introduce the notation and many of the methods of investigation. Part II discusses the precession and nutation of the figure axis and angular momentum vector of the charged top when released initially with zero precession and nutation velocities. Finally, Part III considers the initial precession velocity needed to insure uniform precession without nutation.

I. THE HEAVY TOP IN A GRAVITATIONAL FIELD

Consider a symmetrical rigid body of mass M fixed at one point on the figure axis at a distance l from the center of gravity and placed in a uniform gravitational field. The orientation of the top is described by the usual Euler angles, θ giving the inclination of the figure axis to the vertical, φ the azimuth of the figure axis about the vertical, and ψ the rotation angle of the rigid body about its figure axis. Thus φ corresponds to the precessional motion of the top, while the change in θ describes the nutation of the symmetry axis. It will be assumed that at $t=0$ the top is spinning about its figure axis and is released without any initial precession or nutation velocity.

It will be sufficient for our needs merely to indicate the methods used and to quote the re-

* AEC Postdoctoral Fellow.

sults; details will be found in most texts on analytical mechanics.[1] The total energy is obviously one of the constants of the motion, and is given by

$$E=\tfrac{1}{2}I_1(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)+\tfrac{1}{2}I_3\omega_3^2+Mgl\cos\theta, \quad (1)$$

where I_3 and ω_3 represent, respectively, the principal moment of inertia and the component of the angular velocity along the symmetry axis, and I_1 is the moment of inertia about any axis perpendicular to the symmetry axis.[2] For the given initial conditions the first term in Eq. (1) is zero at $t=0$ and can only increase as the top begins to precess and nutate. On the other hand, the second term, which is proportional to the angular momentum about the figure axis, must be a constant of the motion. This can be seen formally from an examination of the Lagrangian:

$$L=\tfrac{1}{2}I_1(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)+\tfrac{1}{2}I_3(\dot{\psi}+\dot{\varphi}\cos\theta)^2-Mgl\cos\theta. \quad (2)$$

Both ψ and φ are cyclic in the Lagrangian, as they must be from the symmetry of the problem since there is no change if the body is rotated about the vertical or about its figure axis. Hence the corresponding conjugate momenta must be constant, indicating that the components of the angular momentum along the vertical and the figure axis must be conserved:

$$p_\psi\equiv I_1a=I_3(\dot{\psi}+\dot{\varphi}\cos\theta)=I_3\omega_3=L_3, \quad (3)$$

$$p_\varphi\equiv I_1b=(I_1\sin^2\theta+I_3\cos^2\theta)\dot{\varphi}+I_3\dot{\psi}\cos\theta=L_z. \quad (4)$$

For convenience the two momenta have here been set equal to I_1a and I_1b, respectively, where a and b are constants. Since ω_3 is thus conserved, energy can be conserved only if the increase in the first term in Eq. (1) is counterbalanced by a corresponding decrease in the last term, the gravitational potential energy of the top. The kinetic energy of precession and nutation is therefore obtained by the system from the gravitational field. A decrease in the potential energy can arise only if θ increases; hence with the given initial conditions the top always nutates, and in such a manner that the figure axis always falls below its initial position and can never rise above it.

The extent of the nutation can be found directly from the energy equation. Eliminating $\dot{\varphi}$ by means of Eqs. (3) and (4) we can write Eq. (1) in terms of θ alone:

$$\dot{u}^2=(\alpha-\beta u)(1-u^2)-(b-au)^2, \quad (5)$$

where

$$u=\cos\theta,$$
$$\alpha=(2/I_1)(E-\tfrac{1}{2}I_3\omega_3^2) \quad \text{and} \quad \beta=2Mgl/I_1.$$

The expression on the right of Eq. (5) is a cubic in u, whose roots correspond to the zeros of $\dot{\theta}$, and define the limits of the nutation. For the given initial conditions one of the roots must correspond to the initial value of θ, the only other root for real angles lying below this value. The value of the lower limit is most easily obtained for the important special case that the potential energy available for conversion into kinetic energy of precession and nutation shall be much smaller than the initial kinetic energy of the top. This approximation is equivalent to the condition

$$Mgl\ll\tfrac{1}{2}I_3\omega_3^2. \quad (6)$$

Under these circumstances the extent of the nutation is given by

$$\cos\theta_0-\cos\theta_1=\frac{I_1}{I_3}\left(\frac{2Mgl}{I_3\omega_3^2}\right)\sin^2\theta_0. \quad (7)$$

The figure axis oscillates sinusoidally between these limits with a frequency

$$a\equiv(I_3/I_1)\omega_3. \quad (8)$$

Along with the nutation, the figure axis precesses at a velocity which can be found by solving Eqs. (3) and (4) for $\dot{\varphi}$ in terms of θ. Of course the precession cannot be uniform, since $\dot{\varphi}$ must initially be zero, but $\dot{\varphi}$ oscillates sinusoidally about an average precession frequency given by

$$\dot{\varphi}_{Av}=Mgl/I_3\omega_3. \quad (9)$$

Thus, as the initial angular frequency of the top, ω_3, is increased, the extent of the nutation goes down as $1/\omega_3^2$, but the frequency of nutation goes up as ω_3. At the same time the average precession frequency decreases as $1/\omega_3$—the faster the top is spinning about its axis, the slower does it precess. For very high values of ω_3 the nutation is so small that in practice friction at the pivots

[1] See, for example Herbert Goldstein, *Classical Mechanics* (Addison-Wesley, Cambridge, 1950), Secs. 5–7.

[2] Numerical subscripts 1 to 3 refer to the principal axes fixed in the body; letter subscripts xyz denote the space axes, the vertical being the z-axis.

is sufficient to damp it out, and the top *appears* to precess uniformly—the so-called *pseudoregular precession* of Klein and Sommerfeld.

The angular momentum vector **L** shares in the nutation of the figure axis. The vertical component of the angular momentum, given by Eq. (4), is constant, but the magnitude of the angular momentum is a function of θ. This can be seen by comparing L^2,

$$L^2 = I_1^2(\omega_1^2+\omega_2^2)+I_3^2\omega_3^2,$$

with the total energy,

$$E=\tfrac{1}{2}I_1(\omega_1^2+\omega_2^2)+\tfrac{1}{2}I_3\omega_3^2+Mgl\cos\theta.$$

Combining the two equations, L^2 may be written as

$$L^2=2I_1(E-Mgl\cos\theta)+I_3(I_3-I_1)\omega_3^2,$$

which is a linear function of $\cos\theta$. Hence the angular momentum vector must nutate along with figure axis, but in such a manner that L_z remains constant. However, the extent of the nutation will not be the same in the two cases. Denoting the inclination angle of **L** to the vertical by Θ, then to the approximation of Eq. (6) the magnitude of the nutation is given by

$$\cos\Theta_1-\cos\theta_0=-2\left(\frac{I_1}{I_3}\frac{Mgl}{I_3\omega_3^2}\right)^2\sin^2\theta_0\cos\theta_0 \quad (10)$$

(the initial value of Θ being θ_0). The nutation of **L** thus goes as the square of the figure axis nutation, and is therefore always much less than that of the figure axis. Nevertheless, it should be noted that the angular momentum always nutates if the figure axis does.

Finally, it may be noted that the average increase of kinetic energy due to nutation in the approximation of small nutation is given by

$$\tfrac{1}{2}I_1\dot{\theta}_N^2=\frac{Mgl}{8}\left(\frac{I_1}{I_3}\right)\left(\frac{2Mgl}{I_3\omega_3^2}\right)\sin^2\theta_0. \quad (11)$$

Thus the amount of kinetic energy needed for nutation goes down as $1/\omega_3^2$. The average increase of kinetic energy due to precession is just three times that for nutation.

II. THE CHARGED TOP WITH NO INITIAL PRECESSION OR NUTATION VELOCITY

1. Statement of the Problem

A rigid symmetrical body is given an electrical charge density proportional to its mass density, so that there is a constant e/m ratio. It is placed in a constant uniform magnetic field whose directions is chosen as the z-axis of the spatial coordinate system. The body is assumed to have no other forces acting on it, and hence the center of mass can be considered as fixed in space.

The top is initially set spinning with an angular frequency ω_3 about its figure axis. At time $t=0$ the figure axis is released with $\dot{\theta}$ and $\dot{\varphi}$ initially zero. The following section discusses the subsequent motion of the figure axis, and the final section of Part II considers the behavior of the angular momentum vector.

2. Precession and Nutation of the Figure Axis

The Hamiltonian is obviously a constant of the motion, since none of the quantities involved depends on time explicitly. When the forces acting on the system are purely electromagnetic the Hamiltonian has the form

$$H=T+\int\rho\varphi dV,$$

where T is the kinetic energy, φ is the scalar potential, and ρ is the volume charge density. Since only a static magnetic field is assumed present the scalar potential vanishes, and the Hamiltonian reduces to the kinetic energy, which is therefore a constant of the motion. The conservation of the kinetic energy can also be seen in a less formal manner by recalling that a static magnetic field cannot do work on moving charges and therefore the kinetic energy must remain unchanged.

As in the case of the gravitational top the kinetic energy has the form

$$T=(I_1/2)(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)+(I_3/2)\omega_3^2. \quad (12)$$

Again, the first term on the right is initially zero and can only increase in value as the body precesses or nutates. It might also appear at first sight that ω_3 must be conserved, as in the gravitational case, for the symmetry of the system with respect to rotation of the body about its figure axis has not been affected. It is clear from the energy equation, however, that ω_3 cannot be conserved, for otherwise it would not be possible to keep T constant. Indeed, while ψ *is* a cyclic coordinate in the Lagrangian, the conserved conjugate canonical momentum is no longer the

corresponding component of the angular momentum, for now the forces are velocity dependent. Hence ω_3 need not be conserved.

The Lagrangian of the system is given by

$$L = T + \int (\rho/c)\mathbf{v}\cdot\mathbf{A}dV,$$

where **A** is the vector potential, or equivalently by

$$L = T + (e/mc)\int \mu\mathbf{v}\cdot\mathbf{A}dV, \tag{13}$$

where μ is the mass density. As the magnetic field is assumed uniform the vector potential has the form

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B}\times\mathbf{r},$$

where **B** is the constant magnetic intensity vector. Substituting this expression in Eq. (13) and rearranging the triple dot product, the Lagrangian becomes

$$L = T + (e\mathbf{B}/2mc)\cdot\int \mathbf{r}\times\mu\mathbf{v}dV = T - \mathbf{\Omega}\cdot\mathbf{L}, \tag{14}$$

where $\mathbf{\Omega}$ is the Larmor precession vector defined by

$$\mathbf{\Omega} = -e\mathbf{B}/2mc. \tag{15}$$

The second term in the Lagrangian can easily be expressed in terms of the Euler angles of the body by reading it as minus the magnitude of the Larmor frequency times L_z, the component of the angular momentum along the z-axis, given by Eq. (4). The complete expression for the Lagrangian is therefore

$$L = \tfrac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \tfrac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2 \\ - \Omega\dot{\varphi}(I_1\sin^2\theta + I_3\cos^2\theta) - \Omega\dot{\varphi}I_3\cos\theta. \tag{16}$$

Note that ψ and φ are again cyclic coordinates, as is required by the system symmetry, and therefore the conjugate canonical momenta are constants of the motion; but because the potential terms in the Lagrangian are velocity dependent, these canonical momenta cannot be identified with components of the angular momenta. In fact, we have:

$$p_\psi \equiv I_1 a = I_3[\dot{\psi} + (\dot{\varphi} - \Omega)\cos\theta] \\ = I_3\omega_3 - I_3\Omega\cos\theta, \tag{17}$$

$$p_\varphi \equiv I_1 b = (I_1\sin^2\theta + I_3\cos^2\theta)(\dot{\varphi} - \Omega) + I_3\dot{\psi}\cos\theta \\ = L_z - \Omega(I_1\sin^2\theta + I_3\cos^2\theta). \tag{18}$$

Thus, it is not ω_3 which is conserved, but rather ω_3 minus the projection of Ω on the figure axis.

We can now trace the manner in which the kinetic energy is conserved, and incidentally observe the mechanism producing the nutation of the figure axis. As the top begins to precess or nutate it acquires a kinetic energy of precession and nutation. This energy cannot be obtained from the field, as in the gravitational case, for the field can do no work. Hence the additional kinetic energy of precession can only be acquired through a corresponding sacrifice in the kinetic energy of rotation about the figure axis. *In order to precess or nutate the rotation of the charged top about its axis must slow down.* However this decrease in the magnitude of ω_3 must always be such that $I_1 a$ is conserved. From Eq. (17) the magnitude of ω_3 is given by

$$|\omega_3| = |I_1 a/I_3|\,|1 - (I_3\Omega/I_1 a)\cos\theta|,$$

indicating that any slowing down of the top must be accompanied by a corresponding change in θ, i.e., *the figure axis must nutate.* The direction of nutation will depend upon the sign and magnitude of the ratio $I_3\Omega/I_1 a$. For example, if the ratio is negative and less than unity in magnitude the nutation is such as to decrease θ below the initial value. It will be remembered that for the same initial conditions the figure axis of the gravitational top always nutates below its initial position, i.e., θ increases. These results may be summarized by stating that, to conserve kinetic energy, precession of the top is accompanied by a slowing down of the top's initial spin, and that the figure axis must then in turn nutate in order to conserve the canonical momentum corresponding to ψ.

The extent of the nutation may be found by employing the same methods used for the gravitational top. Equations (17) and (18) may be solved for $\dot{\varphi}$ and $\dot{\psi}$ in terms of θ and the constants a and b:

$$\dot{\varphi} = \Omega + \frac{b - a\cos\theta}{\sin^2\theta}, \tag{19}$$

$$\dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta\frac{b - a\cos\theta}{\sin^2\theta}. \tag{20}$$

These expressions differ from those obtained in the gravitational case only in that $\dot{\varphi}$ is replaced

by $\dot{\varphi}-\Omega$.[3] With the help of Eqs. (19) and (20) the kinetic energy T, Eq. (12), can be expressed solely in terms of θ, $\dot{\theta}$, and the constants of the motion. After some algebraic manipulation one obtains the equation

$$\dot{u}^2=\alpha(1-u^2)-(b-au)^2-\epsilon\Omega^2(1-u^2)^2, \quad (21)$$

corresponding to Eq. (5) for the heavy top. Here u stands for $\cos\theta$, as before, and in addition

$$\alpha=\frac{2T}{I_1}-\Omega b-\frac{I_1a^2}{I_3}-\frac{I_3}{I_1}\Omega^2$$

and

$$\epsilon=(I_1-I_3)/I_1, \quad (22)$$

which is thus a measure of the eccentricity of the inertia ellipsoid. Note that the polynomial on the right in Eq. (21) is now a quartic in u. According to the initial conditions $\dot{\varphi}$ is zero when $u=u_0$, and by Eq. (19) a and b are therefore connected by the relation

$$b=au_0-\Omega(1-u_0^2). \quad (23)$$

The initial conditions further require that u_0 be one of the roots of Eq. (21) which implies that α must have the value

$$\alpha=\Omega^2(1-u_0^2)(1+\epsilon). \quad (24)$$

With these substitutions Eq. (21) can be written in the following form, explicitly showing that u_0 is one of the roots:

$$\dot{u}^2=(u_0-u)a^2\{2(\Omega/a)(1-u_0^2)-(u_0-u)$$
$$+(\Omega/a)^2(u_0+u)(1-\epsilon-u_0^2+\epsilon u^2)\}. \quad (25)$$

We shall be interested primarily in the case in which the ratio Ω/a is a small number much less than one. Physically this means that the Larmor frequency shall be much smaller than the initial frequency of the top about its axis, a condition which is abundantly satisfied in most of the applications of interest to atomic physics. Under this approximation the terms in $(\Omega/a)^2$ in Eq. (25) may be dropped, and the expression for $\dot{u}^2$ reduces to

$$\dot{u}^2=a^2(u_0-u)\{2(\Omega/a)(1-u_0^2)-(u_0-u)\}, \quad (26)$$

correct to first order in Ω/a. The extent of the nutation will be given by $x_1=u_0-u_1$, where u_1 is the root of the quantity in curly brackets in Eq. (26):

$$x_1=\cos\theta_0-\theta_1=2(\Omega/a)\sin^2\theta_0$$
$$\approx 2(I_1\Omega/I_3\omega_3)\sin^2\theta_0. \quad (27)$$

Thus the amplitude of the nutation decreases the faster the initial frequency of the top, but the decrease is slower than for the gravitational top, going only as a^{-1} [see Eq. (7)].

If x denotes the variable u_0-u, then Eq. (26) can be rewritten as

$$\dot{x}^2=a^2x(x_1-x),$$

which has the solution

$$x=\tfrac{1}{2}x_1(1-\cos at). \quad (28)$$

Just as in the gravitational case the *frequency* of nutation is again

$$a\approx I_3\omega_3/I_1,$$

which increases the faster the top is spun initially. From Eq. (23) it is seen that the precession frequency $\dot{\varphi}$, Eq. (19), can be written as

$$\dot{\varphi}=ax/\sin^2\theta_0,$$

or by Eq. (28) as

$$\dot{\varphi}=(ax_1/2\sin^2\theta)(1-\cos at)=\Omega(1-\cos at).$$

Thus $\dot{\varphi}$ is not constant but oscillates about an average precession frequency equal to the Larmor frequency:

$$\dot{\varphi}_{Av}=\Omega, \quad (29)$$

which is the expected result. (It should be kept in mind, however, that Eq. (29) is only correct to first order in Ω/a.) Whereas the conclusion that the average precession frequency is given by Ω will occasion no surprise, it may be noted that this result differs markedly from the gravitational case. There the average frequency of precession was not constant but decreased inversely as a.

One consequence of the independence of $\dot{\varphi}_{Av}$ on a is that the kinetic energy which is associated with the precession does not decrease as a is increased. The same conclusion holds for the kinetic energy of nutation, for although the amplitude of nutation goes down as a^{-1} the frequency goes up as a. In fact the average decrease in kinetic energy of rotation about the figure axis due to precession and nutation is

[3] See reference 1, Eqs. (5–50) and (5–51).

given by:

$$\Delta(\tfrac{1}{2}I_3\omega_3^2)_N \cong I_3\omega_3(\Delta\omega_3)_N \cong I_1 a\Omega\Delta u = \tfrac{1}{2}I_1 a\Omega x_1 = I_1\Omega^2 \sin^2\theta_0,$$

to lowest order in (Ω/a). Of this amount one quarter arises from the nutation:

$$\tfrac{1}{2}I_1\dot{\theta}_N^2 = \frac{I_1\Omega^2}{4}\sin^2\theta_0,$$

while the rest can be ascribed to the precessional motion:

$$\tfrac{1}{2}I_1\dot{\varphi}_N^2 \sin^2\theta_0 = (3I_1\Omega^2/4)\sin^2\theta_0.$$

By increasing a (or equivalently, the initial value of ω_3) one can reduce the amplitude of the nutation below any desired limit. However the effects of the nutation can never be made negligible, for the amount of kinetic energy which has to be devoted to nutational motion remains constant and is always a constant fraction of the energy of precession.

III. THE CONDITIONS FOR UNIFORM PRECESSION

The motion of the angular momentum vector, **L**, may be examined in a manner similar to that used for the gravitational top. Here the z-component (i.e., along **B**) is no longer constant, but is a function of θ according to Eq. (18):

$$L_z = I_1 b + \Omega(I_1 \sin^2\theta + I_3\cos^2\theta) = I_1(b+\Omega) - \Omega\epsilon I_1 \cos^2\theta. \quad (30)$$

The magnitude of **L** is likewise a function of the nutation angle, for L^2 is given by

$$L^2 = I_1^2(\omega_1^2+\omega_2^2) + I_3^2\omega_3^2 = 2I_1 T - I_3(I_1 - I_3)\omega_3^2,$$

or

$$L^2 = 2I_1T - (\epsilon I_1/I_3)(I_1 a + \Omega I_3 \cos\theta)^2, \quad (31)$$

using Eq. (17) for ω_3. If the ratio Ω/a is small the results of the last section for the extent of the nutation of the figure axis may be applied to Eqs. (30) and (31) to obtain the amplitude of the nutation of **L** to the same order of approximation. If Θ is used to denote the angle of L to the magnetic field, then straightforward algebraic manipulation shows that Θ is given by:

$$\cos\Theta = \cos\theta_0[1 + (\Omega\epsilon/a)(\cos\theta - \cos\theta_0)]$$

to the lowest order in Ω/a. The extent of the angular momentum nutation, by Eq. (27), is therefore

$$\cos\Theta - \cos\theta_0 = (\Omega/a)\epsilon x_1 \cos\theta_0 = 2(\Omega/a)^2\epsilon \sin^2\theta_0 \cos\theta_0, \quad (32)$$

again to the lowest nonvanishing order in Ω/a. As in the gravitational top the nutation of **L** goes as the square of the figure axis nutation, and is therefore always much smaller. One curious feature of Eq. (32) is that the nutation of **L** vanishes when $\epsilon = 0$, i.e., when the inertia ellipsoid is a sphere. This conclusion is not the result of any of the approximations made in obtaining Eq. (32) for it can be seen from Eqs. (30) and (31) that both L_z and L^2 are constant if ϵ is zero.

It might also appear as unexpected that *any* nutation of the angular momentum is obtained, for a frequently encountered elementary derivation seems to indicate that **L** precesses uniformly at all times. In this derivation, the torque on the spinning top due to the magnetic field is given as

$$\mathbf{N} = \mathbf{M}\times\mathbf{B} = \mathbf{\Omega}\times\mathbf{L}, \quad (33)$$

where **M** is the magnetic moment. The torque however, is the time rate of change of the angular momentum, and therefore

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \mathbf{\Omega}\times\mathbf{L}. \quad (34)$$

Equation (34) states that **L** is a vector of constant magnitude which precesses uniformly about the direction of **B** with the precession frequency **Ω**. There appear to be no approximations in this result, whereas Eq. (32) implies that uniform precession cannot take place unless $\epsilon = 0$.

The resolution of the paradox lies in the torque equation (33). Customarily it is obtained from the fact that the potential of a magnetic dipole has the form

$$\mathbf{M}\cdot\mathbf{B} = MB\cos\Theta.$$

The derivative of this expression with respect to Θ is then taken as $MB\sin\Theta$, which corresponds to $\mathbf{M}\times\mathbf{B}$. This is perfectly correct for a permanent magnetic dipole. Here, however, the magnitude of the magnetic moment is proportional to the angular momentum, which is in general a function of the position of the top. Only when the inertia ellipsoid is a sphere is the magnitude of the angular momentum, and there-

fore of the magnetic moment, a constant independent of position. Hence the torque on the spinning charged top in a magnetic field is not given completely by $\mathbf{M}\times\mathbf{B}$; there are in addition terms which arise from the change in the magnitude of the magnetic moment as the body moves.

It is of some interest to obtain the explicit form for the correct torque acting on the top. Fundamentally the torque must be given by

$$\mathbf{N}=\int \mathbf{r}\times(\rho/c)(\mathbf{v}\times\mathbf{B})dV$$

$$=(e/mc)\int \mu\mathbf{r}\times(\mathbf{v}\times\mathbf{B})dV, \quad (35)$$

where, as before, $\mathbf{r}$ is the radius vector from the center of mass, ρ is the charge density, and μ is the mass density. Expanding the triple cross product the torque becomes:

$$\mathbf{N}=(e/mc)\int \mu[\mathbf{v}(\mathbf{r}\cdot\mathbf{B})-\mathbf{B}(\mathbf{r}\cdot\mathbf{v})]dV.$$

The second term in the brackets must vanish, for since the body is rotating about a fixed point $\mathbf{r}$ is perpendicular to $\mathbf{v}$. By adding and subtracting a term of the form $\mathbf{r}(\mathbf{v}\cdot\mathbf{B})$ the torque can now be written:

$$\mathbf{N}=(e/2mc)\int \mu[\mathbf{v}(\mathbf{r}\cdot\mathbf{B})-\mathbf{r}(\mathbf{v}\cdot\mathbf{B})]dV$$

$$+(e/2mc)\int \mu[\mathbf{v}(\mathbf{r}\cdot\mathbf{B})+\mathbf{r}(\mathbf{v}\cdot\mathbf{B})]dV. \quad (36)$$

The expression in the first integral in Eq. (36) will be recognized as the expansion of a triple cross product, and in fact this integral is

$$(e/2mc)\int dV\mathbf{B}\times(\mu\mathbf{v}\times\mathbf{r})$$

$$=(e/2mc)\int (\mathbf{r}\times\mu\mathbf{v})dV\times\mathbf{B}$$

or

$$\mathbf{M}\times\mathbf{B}.$$

The correction terms in the torque to $\mathbf{M}\times\mathbf{B}$ must therefore come from the second integral in Eq. (36). Remembering now that $\mathbf{v}$ is given by

$$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r},$$

the expression in the brackets in the second integral can be put in the form

$$\boldsymbol{\omega}\times\mathbf{rr}\cdot\mathbf{B}-\mathbf{rr}\cdot\boldsymbol{\omega}\times\mathbf{B}, \quad (37)$$

where the elements of the triple dot product have been permuted. Both terms in the expression (37) are in the form of a dyad $\mathbf{rr}$ operated upon by a combination of vectors. If $\mathbf{rr}$ is replaced by the unit dyadic $\mathbf{1}$ the expression vanishes, for then Eq. (37) becomes

$$\boldsymbol{\omega}\times\mathbf{1}\cdot\mathbf{B}-\mathbf{1}\cdot\boldsymbol{\omega}\times\mathbf{B}=\boldsymbol{\omega}\times\mathbf{B}-\boldsymbol{\omega}\times\mathbf{B}=0.$$

We are therefore at liberty to rewrite Eq. (37) as

$$-\boldsymbol{\omega}\times(\mathbf{1}r^2-\mathbf{rr})\cdot\mathbf{B}+(\mathbf{1}r^2-\mathbf{rr})\cdot\boldsymbol{\omega}\times\mathbf{B},$$

where the combination $\mathbf{1}r^2-\mathbf{rr}$, it will be recalled, appears in the definition of the inertia tensor.[4] In fact, the second integral in the torque equation (36) can now be put in the form

$$-(e/2mc)\boldsymbol{\omega}\times\int \mu(\mathbf{1}r^2-\mathbf{rr})dV$$

$$+(e/2mc)\int \mu(\mathbf{1}r^2-\mathbf{rr})dV\cdot\boldsymbol{\omega}\times\mathbf{B}$$

$$=\boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\Omega}-\mathbf{I}\cdot\boldsymbol{\omega}\times\boldsymbol{\Omega},$$

where $\mathbf{I}$ is the inertia tensor (or dyadic). The complete expression for the torque is therefore:

$$\mathbf{N}=\mathbf{M}\times\mathbf{B}+\boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\Omega}-\mathbf{I}\cdot\boldsymbol{\omega}\times\boldsymbol{\Omega}, \quad (38)$$

or equivalently:

$$\mathbf{N}=\boldsymbol{\Omega}\times\mathbf{I}\cdot\boldsymbol{\omega}+\boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\Omega}-\mathbf{I}\cdot\boldsymbol{\omega}\times\boldsymbol{\Omega}. \quad (39)$$

If the inertia ellipsoid is a sphere $\mathbf{I}$ is then proportional to the unit dyadic $\mathbf{1}$ and the correction terms in the torque vanish, as predicted beforehand. For a symmetrical rigid body these correction terms, resolved along the principal axes, appear as

$$-\omega_2\Omega_3\epsilon I_1\mathbf{i}+\omega_1\Omega_3\epsilon I_1\mathbf{j}+(\omega_1\Omega_2-\omega_2\Omega_1)\epsilon I_1\mathbf{k}. \quad (40)$$

The considerations of Part II have all assumed the initial conditions of zero precession and nutation velocities. The question arises whether it is possible to start the top off with an initial precession in such a manner that it continues to precess uniformly. However, before obtaining

[4] Reference 1, p. 149.

this value of the initial $\dot{\varphi}$ for which uniform precession occurs, it will be necessary to make some remarks about Larmor's theorem.

In its most general form Larmor's theorem states that the effect of a magnetic field on a system composed of particles with the same e/m ratio is equivalent to transferring the system to moving coordinate axes which rotate with the Larmor frequency. As is well known, the Larmor theorem in this form is only approximate, for it neglects terms of the order of B^2.[5] The extent to which Larmor's theorem is fulfilled in the case of a rigid body is easiest seen by examining the form of the Lagrangian, Eq. (14):

$$L=\tfrac{1}{2}I\omega^2-\mathbf{\Omega}\cdot\mathbf{L}=\tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega}-\mathbf{\Omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega}. \tag{41}$$

In a set of axes rotating with the frequency $\mathbf{\Omega}$ (which will be designated as the Larmor axes) the apparent angular velocity of the body is

$$\boldsymbol{\omega}'=\boldsymbol{\omega}-\mathbf{\Omega},$$

and, remembering the symmetry of $\mathbf{I}$, the Lagrangian appears as

$$L=\tfrac{1}{2}\boldsymbol{\omega}'\cdot\mathbf{I}\cdot\boldsymbol{\omega}'-\tfrac{1}{2}\mathbf{\Omega}\cdot\mathbf{I}\cdot\mathbf{\Omega}. \tag{42}$$

Terms linear in Ω no longer appear in the Lagrangian, which would correspond exactly to the Lagrangian of a free particle in the Larmor axes except for the presence of the last term. It is this term in Ω^2 which makes the Larmor theorem only an approximate statement, correct only to first order in

$$\Omega/\omega\approx\Omega/a.$$

The Lagrangian may also be written in the form

$$L=\tfrac{1}{2}I_\omega\omega'^2-\tfrac{1}{2}I_\Omega\Omega^2, \tag{43}$$

where I_ω and I_Ω are the moments of inertia along $\boldsymbol{\omega}'$ and $\mathbf{\Omega}$, respectively. If the inertia ellipsoid is a sphere then all moments of inertia are equal and the term in Ω^2 in Eq. (43) is a constant independent of the motion of the body. As such it has no effect on the equations of motion and may be dropped from the Lagrangian. Thus, Larmor's theorem is rigorously true for a sphere.

It must not be thought that the nutations and nonuniform precessions found in the preceding Part stem from the approximate nature of the Larmor theorem. One can obtain a Lagrangian for which the theorem is rigorously true, simply by adding the offending Ω^2 term to L:

$$L'=T-\mathbf{\Omega}\cdot\mathbf{L}+\tfrac{1}{2}I_\Omega\Omega^2. \tag{44}$$

If this be taken as the Lagrangian for the charged top then it will be readily seen that there will be no change in the definitions of the canonical momenta, for the new term does not involve any of the angular velocities; nor will there be any change in the formulas obtained for the figure axis nutation and precession, since all terms in $(\Omega/a)^2$ had already been omitted in their derivation. The nutation of the angular momentum vector is likewise unaffected by any term in the Lagrangian involving Ω^2. The origin of the nutation is clear if it is remembered that the initial condition $\dot{\varphi}=0$ in the space axes corresponds to an initial condition $\dot{\varphi}'=-\Omega$ in the Larmor axes. The observed nutation and nonuniform precession in the space axes thus appears as the Poinsot precession of the free top in the Larmor axes.

On the other hand the effects of the Ω^2 term in the Lagrangian show up most directly in considering the rigorous uniform precession frequency. It is clear from the previous discussion that this will not, in general, be exactly equal to the Larmor frequency, that there will usually be additional terms of the order of $\Omega(\Omega/a)$ or higher. One can also predict that in the special case the inertia ellipsoid is spherical the uniform precession frequency *will* reduce to the Larmor frequency.

A formal solution for the uniform precession frequency can be obtained from the energy equation for the top in its one-dimensional form, Eq. (21). The initial conditions to be imposed now are that at $t=0$

$$\dot{\theta}=0,\quad\text{and}\quad \dot{\varphi}_0=\Omega+a\delta,$$

where δ is to be determined such that $\dot{\theta}$ remains zero, and there is uniform precession without nutation. An equivalent form for the condition to be satisfied by δ is that u_0 be a double root of Eq. (21). With these boundary conditions Eq. (19) states that b and a are connected by the relation

$$b=au_0+a\delta(1-u_0^2)$$

and the requirement that $\dot{u}$ vanish at $t=0$ leads

[5] J. H. Van Vleck, *Electric and Magnetic Susceptibilities* (Oxford University Press, London, 1932), Sec. 8

to the value for α of

$$\alpha=(a^2\delta^2+\epsilon\Omega^2)(1-u_0^2).$$

With these conditions satisfied, Eq. (21) can be written in the form

$$\dot{u}^2=(u_0-u)[\{a^2\delta^2(1-u_0^2)-\epsilon\Omega^2(1-u^2)\}(u_0+u) -a^2(u_0-u)-2a^2\delta(1-u_0^2)]. \quad (45)$$

For u_0 to be a double root of Eq. (45), δ must be such that the expression in the brackets vanishes when $u=u_0$. Consequently δ satisfies a quadratic equation given by

$$(a^2\delta^2-\epsilon\Omega^2)u_0-a^2\delta=0. \quad (46)$$

The exact solutions for Eq. (46) are simply

$$\delta=\frac{1\pm(1+4u_0^2\epsilon(\Omega/a)^2)^{\frac{1}{2}}}{2u_0}. \quad (47)$$

For small values of the ratio Ω/a one of the solutions represented by Eq. (47) is quite small, as we would expect; the other solution, however, is not small but is of the order of $1/u_0$. This latter case corresponds to the "fast precession" of the heavy top, in which the angular momentum along the figure axis is composed almost entirely of the component of the initial precessional angular momentum given to the top. Here, it will be noted, the contribution to the angular velocity ω_3 arising from this initial value of $\dot{\varphi}$, neglecting all terms in Ω, is:

$$\dot{\varphi}_0\cos\theta\simeq a\delta\cos\theta\simeq a.$$

Thus most of a comes from the initial precession given the top, little from the initial rotation of the top about its figure axis. Superimposed on this larger precession there is still the Larmor precession given in first order by Ω. Since it is the Larmor precession which is of primary interest to us we may disregard this solution of Eq. (46) and consider only the other solution, which corresponds to the "slow precession."

When the ratio Ω/a is small the lowest solution given by Eq. (47) is

$$\delta=-u_0\epsilon(\Omega/a)^2.$$

The uniform precession frequency, accurate to terms in $(\Omega/a)^2$ is therefore:

$$\dot{\varphi}=\Omega[1-\cos\theta_0\epsilon(\Omega/a)]. \quad (48)$$

In agreement with our qualitative considerations it is seen that the uniform precession frequency is equal to the Larmor frequency when the inertia ellipsoid is a sphere and in general differs from Ω by terms of no lower order than $(\Omega/a)^2$.

The precession frequency given by Eq. (48) may be obtained by another method, which makes use of the Euler equations of motion. Taking, for example, the 1-component equation, we have:

$$I_1(\dot{\omega}_1-\epsilon\omega_2\omega_3)=N_1 \quad (49)$$

or, substituting the form of the torque as given by Eqs. (39) and (40):

$$I_1(\dot{\omega}_1-\epsilon\omega_2\omega_3)=\omega_3I_3\Omega_2-\omega_2I_1\Omega_3-\epsilon\omega_2\Omega_3I_1. \quad (50)$$

The uniform precession frequency can now be found by expressing the angular frequencies and components of $\mathbf{\Omega}$ in terms of the Euler angles, imposing the conditions $\dot{\varphi}$, $\dot{\psi}$ constant, $\dot{\theta}=0$ (i.e., uniform precession, no nutation), and finding the value of $\dot{\varphi}$ for which Eq. (50) is then satisfied. Carrying out this program, the uniform precession frequency is given as the solution of the equation

$$(\dot{\varphi}-\Omega)[I_3(\dot{\varphi}\cos\theta_0+\dot{\psi})\sin\theta_0-I_1\dot{\varphi}\sin\theta_0\cos\theta_0] =-\epsilon\dot{\varphi}I_1\Omega\sin\theta_0\cos\theta_0. \quad (51)$$

If only the first correction terms to the Larmor frequency are of interest, then $\dot{\varphi}$ on the right of Eq. (51) may be set equal to Ω, while all terms involving $\dot{\varphi}$ in the brackets may be neglected compared to the $\dot{\psi}$ term. To the same approximation $I_3\dot{\psi}$ can be replaced by I_1a, and Eq. (51) has the solution

$$\dot{\varphi}-\Omega=-\epsilon\Omega(\Omega/a)\cos\theta_0,$$

which agrees with Eq. (48). Needless to say the same result would have been obtained if we had started from the 2-component of the Euler equations. The 3-equation is satisfied automatically if the precession is uniform, no matter what the value of $\dot{\varphi}$.

It is clear that these corrections, assuming they can be applied literally to atomic phenomena, are almost always quite small. For an orbital electron the Larmor frequency even at the highest fields is still in the microwave region, of the order of 10^{10}sec^{-1}, while a presumably corresponds to the atomic term values, usually of the order of 10^{15}sec^{-1}, in the visible or ultraviolet region. The ratio Ω/a under these conditions is

negligibly small. However, if the transitions involve levels very close to the continuous limit, then a (or its quantum analog) can be quite small and the corrections to the uniform precession frequency can become important or even dominant. The quantum shift analogous to Eq. (48) has in fact been observed by Jenkins and Segrè[6] in the Zeeman effect of the lines close to the limit of the principal series of sodium.

Under certain circumstances the Ω^2 terms in the Lagrangian or Hamiltonian can become all important. Thus in ^1S states the linear terms in Ω drop out because the states have no angular momentum and the square terms then contain the total interaction with the magnetic field. As is well known,[7] the term in the Lagrangian (43) involving Ω^2 then leads directly in both classical and quantum mechanics to the standard Pauli-Langevin formula for atomic diamagnetism.

The author acknowledges with thanks his indebtedness to Prof. N. F. Ramsey of Harvard University, who suggested the need for this investigation.

[6] F. A. Jenkins and E. Segrè, Phys. Rev. 55, 52 (1939). See also the succeeding paper by L. I. Schiff and H. S. Snyder on the quantum theory of the effect, Phys. Rev. 55, 59 (1939).

[7] Reference 5, pp. 91, 206.

The Well-Informed Heat Engine

RICHARD C. RAYMOND
The Pennsylvania State College, State College, Pennsylvania

(Received July 21, 1950)

The failure of a heat engine to defeat the second law of thermodynamics by using density fluctuations to convert heat to work without leaving other changes in the universe is usually explained by saying that the fluctuations of the engine itself would defeat such an operation or that the microscopic nature of the fluctuations prevents their being put to a macroscopic use. It is shown here that with a proper definition of stored information, a heat engine can be made to convert heat to work without other changes in its immediate system, provided that an outside observer creates in the system a negative information entropy equal to the negative entropy change involved in the operation of the engine. This equivalence of a communication entropy change to a thermodynamic entropy change leads to the definition of the entropy of a nonequilibrium system as the algebraic sum of the thermodynamic entropy which the system would have at equilibrium and the information entropy necessary to construct the specified state from the equilibrium state.

THE suggestion that a heat engine might be built to convert heat into work without leaving other changes in the universe has been discussed briefly by a number of writers. In a modern textbook[1] on kinetic theory the possibility is dismissed by the statement that the second law of thermodynamics applies only to normal or average behavior of macroscopic systems, or by the statement that any such machine would itself be subject to fluctuations which would prevent its successful operation. Fluctuations of appreciable magnitude are apparent in a case such as that of Brownian motion, and it seems difficult to imagine that any set of fluctuations could cancel any other set of fluctuations for any appreciable period of time. It is, therefore, of interest to examine a theoretical heat engine which operates on the basis of density fluctuations in a gas to determine in what way changes must be left in the universe as a result of its operation. It will be interesting also in this connection to examine a case of physical relationship between the information entropy defined by Hartley,[2] Shannon,[3] and Wiener[4] and the common thermodynamic entropy.

The Engine

Figure 1 shows the heat engine which will be discussed, and Fig. 2 illustrates its cycle of

[1] E. H. Kennard, *Kinetic Theory of Gases* (McGraw-Hill Book Company, Inc., New York, 1938), p. 368.

[2] R. V. L. Hartley, Bell System Tech. J. 7, 535 (1928).

[3] C. E. Shannon, Bell System Tech. J. 27, 379 (1948).

[4] N. Wiener, *Cybernetics* (John Wiley & Sons, Inc., New York, 1948).

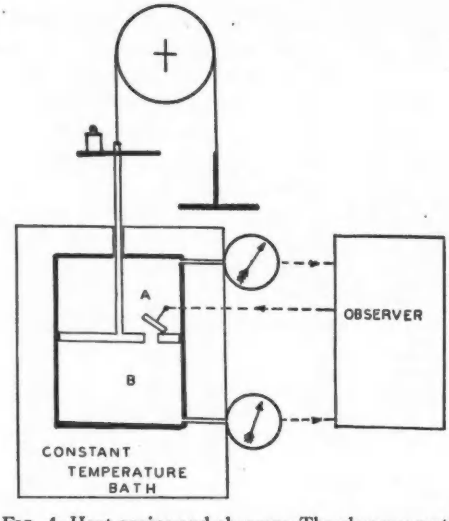

FIG. 1. Heat engine and observer. The observer watches the gauges and controls the gate in the piston to operate the engine through a cycle in which heat is absorbed and work is done without leaving other changes in the system which does not include the observer.

operation. The engine consists of a cylinder fitted with a piston which divides the total volume of the cylinder into an upper part *A* and a lower part *B*. The piston is connected with a piston rod of negligible diameter which moves weights and weight pans in such a way as to do mechanical work. The cylinder is fitted with pressure gauges at both ends and surrounded by a constant temperature bath. The piston is provided with a gate under the control of an external observer who manipulates the gate in response to the time average readings of the pressure gauges in order to carry out the desired thermodynamic cycle. The system to be considered in the computations includes the cylinder, piston, weight pans and weights and the constant temperature bath but does not include the observer. This choice of system boundaries makes it possible to compute the contribution of the observer to the system in terms of negative information entropy and makes it unnecessary to consider in detail the processes of the observer.

The operating cycle of the engine is diagrammed in Fig. 2, which shows the pressure in part *A* of the cylinder plotted against the volume of part *A* as the cycle of operation is carried out. The cycle starts at *O*. In process *OQ* the following steps are carried out: (a) the piston is locked in place so that the volume cannot change, and the gate is opened; (b) after a time interval sufficient to make the distribution of molecules of gas between parts *A* and *B* random with respect to the initial distribution the gate is closed, and time average readings of the two pressure gauges are taken over intervals sufficiently long to insure accurate indication of the distribution of gas molecules between the two chambers; (c) if point *Q* has been attained, the gate is locked, and the piston is loaded and released for a reversible isothermal expansion through process *QX*. If point *Q* has not been reached, the piston is kept in position, and the gate is opened for a time interval sufficient to insure a new random distribution and then closed again as in (b) above. No heat is exchanged with the bath in process *OQ*, but the entropy of the system is reduced through the storage of information in the system by the observer.

In process *QX* heat is added to the cylinder from the bath, and an equivalent amount of work is done. At point *X* the piston is again locked in position and the gate is opened. Steps similar to that in process *OQ* are repeated until point *Y* is reached. Again the entropy of the system is reduced through the storage of information. In process *YO* the gate is locked, the piston

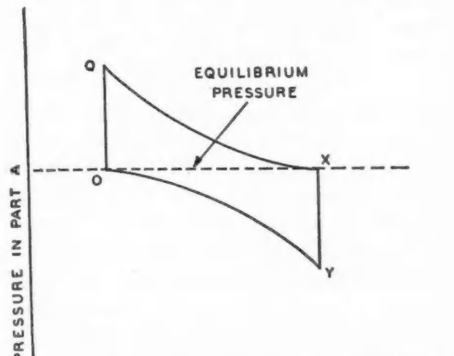

FIG. 2. Operating cycle of the heat engine. In process *OQ* random fluctuations result in increase in pressure in part *A* of the cylinder. Process *QX* is a reversible isothermal expansion of part *A*. Process *XY* is a fluctuation reversing that of process *OQ*, and process *YO* is an isothermal expansion of part *B* which returns the cylinder to its original condition.

is unlocked and a reversible isothermal expansion is carried out, resulting in an absorption of heat from the bath with an equal output of work. At point O the gate is opened, returning the cylinder and piston to its original condition and leaving the system unchanged except for the disappearance of a certain amount of heat and the performance of an equal amount of mechanical work.

The following symbols will be used in the mathematical discussion of the entropy changes involved in the engine cycle.

V_a = initial and final volume of part A,
V_b = initial and final volume of part B,
$V = V_a + V_b$,
N_a = equilibrium number of molecules in part A when the volume is V_a,
N_b = equilibrium number of molecules in part B when the volume is V_b,
$N = N_a + N_b$,
N_e = number of molecules shifted from part B to part A in process OQ,
V_e = increase in volume of the part A during process QX.

If the gas molecules in the cylinder are so chosen that the gas obeys the perfect gas equation of state the work done and the heat absorbed in the two reversible isothermal processes QX and YO are each given by

$$W = N_e kT[\log[(V_a + V_e)/V_a] + \log[V_b/(V_b - V_e)]], \quad (1)$$

where k is Boltzmann's constant and T is the temperature of the bath. If the contribution of the observer in manipulating the engine could be ignored, this would be accompanied by a decrease in the entropy of the system without other changes. We shall now see, however, that with the proper choice of the definition for information stored in the system by the observer we can demonstrate that the negative entropy created in the system by the information storage process is equal to the change in entropy resulting from the work indicated in Eq. (1). We may then make use of a previous result[5] to show that the communication of this information from the observer to the system has resulted in an increase of the entropy of the large system including the observer which is larger than the decrease in the entropy in the small system due to information storage.

The plausibility of the definition of quantity of information to be adopted may be justified in the manner used by Wiener.[4] When the gate is closed in the piston and the piston is locked in position in the cylinder in such a way that the pressures on the two sides of the piston are not equal, then the system is not in equilibrium, and the relationship between the pressures may be taken to indicate an item of information, such as a number, stored in the system. Process OQ, the method of storing the number in the system, will on the average require a number of trials to reach point Q, and the number of digits used to express this number of trials will be proportional to the logarithm of the number. The amount of information stored in the cylinder when it is not in equilibrium will thus be taken as the logarithm of the number of trials which are necessary, on the average, to establish the particular nonequilibrium state. If we designate the nonequilibrium state in terms of N_e, the number of displaced molecules, we may compute the probability of state e and the expected number of trials necessary to establish state e by the simple probability assumptions of the kinetic theory. If the probability that state e exists on any closing of the gate is P_e, the probable number of trials necessary to establish state e is $1/P_e$, and the information content of the system is defined as

$$D_e = \log(1/P_e) = -\log P_e. \quad (2)$$

Assuming that all the molecules are independent, as in a perfect gas, the probability of state e in performing process OQ is

$$P_e = (V_a/V)^{(N_a+N_e)}(V_b/V)^{(N_b-N_e)} N!/(N_a+N_e)!(N_b-N_e)! \quad (3)$$

When the numbers of molecules are large and the deviation from equilibrium is appreciable, the first terms of Stirling's approximation for the factorial may be used to write Eq. (2) as

$$D_e \doteq (N_a+N_e)\log(N_a+N_e) + (N_b-N_e)\log(N_b-N_e) + N\log V - (N_a+N_e)\log V_a - (N_b-N_e)\log V_b - N\log N. \quad (4)$$

If $N_e = 0$, and if $N_a/V_a = N_b/V_b$ the information

[5] R. C. Raymond, Am. Scientist 38, 273 (1950).

storage involved in closing the gate on the system as given by Eq. (4) disappears. If, however, the second term of the Stirling approximation is used, the total is not zero. We have,

$$D_0 \doteq \tfrac{1}{2}\log(2\pi N_a N_b/N). \quad (5)$$

Information is stored in the system at the completion of steps OQ and XY. Some information is lost in the system on opening the gate at the completion of steps QX and YO. The net information stored in the system in the performance of its cycle may be determined by writing Eqs. (4) and (5) in forms appropriate to points Q, X, Y, and O and then taking the sum of information inputs at point Q and Y, less the sum of information losses at points X and O. This process results in

$$D_c = N_e[\log(V_a+V_e)/V_a + \log V_b/(V_b-V_e)]. \quad (6)$$

This result is identical with Eq. (1) except for the factor kT, and it indicates that the information entropy stored in the system during the cycle may be regarded as a negative physical entropy. This point of view suggests further that the entropy of the system in a nonequilibrium state characterized by e may be defined as

$$S = S_{eq} - kD_e, \quad (7)$$

where S_{eq} is the entropy which the system has with the gate open at equilibrium.

The observer has been omitted from the system because it is of interest to consider his contribution to the creation of negative entropy in the system through the storage of information without considering his particular physical processes. No observer yet considered has proved capable of storing information in any system without degrading an amount of energy sufficient to make the total entropy change in a system, including the observer, positive. The second law is therefore not in danger through the treatment of information as a form of negative physical entropy. The use of the entropy definition of Eq. (7) helps in explaining the selective decrease of entropy in some systems in apparent contradiction to the second law, and a careful consideration of the information storage possible in a system assists in distinguishing equilibrium systems from steady state systems. If, for instance, the two sides of the piston in the cylinder of Fig. 1 were filled to the same pressure with identical molecules, the old thermodynamics would predict no change in entropy on the opening of the gate. Classical statistical mechanics would predict a change in entropy, and the wave-mechanical statistical mechanics would predict no change. Information theory shows that the closing of the gate on any predetermined distribution of the molecules requires some number of trials and that the system with the gate closed therefore contains stored information and a lower entropy than the system with the gate open. The total entropy of the sum of two masses of identical gas at constant temperature and pressure is therefore larger than the sum of the entropies of the separate masses under the information theory and the classical statistical mechanics, whereas this inequality does not exist in statistical wave mechanics or in classical thermodynamics.

A Laboratory Experiment on the Determination of γ for Gases by Self-Sustained Oscillations

W. F. KOEHLER
U. S. Naval Postgraduate School, Annapolis, Maryland

(Received July 3, 1950)

The apparatus used is a Rüchardt Apparatus with a small hole bored through the wall of the tube which permits gas to escape when the ball is above the hole. The ball is caused to oscillate symmetrically with respect to the hole by adjusting an inlet valve. When adjusted, the constant rate of inflow of gas is equal to the average rate of escape of gas. The resulting fluctuation of the pressure is sinusoidal to a high degree of approximation and maintains the oscillations of the ball indefinitely. An elementary theory predicts that γ can be determined from the measurement of the period of oscillation, mean pressure, mass of the ball, diameter of the ball, and volume of the container. Results for most of the common gases are reproducible to within 0.2 percent and differ from the accepted values by 1 percent.

ANY teacher familiar with Rüchardt's method of measuring γ for gases appreciates, from a pedagogical point of view, the value of an experiment using the method. The following is a brief discussion of his apparatus and method of measuring γ. The gas under investigation is enclosed in a container with an attached, vertical, precision-bored, glass tube in which a matching steel ball can move freely. His apparatus is similar to that shown in Fig. 1 without the hole in the wall of the vertical tube. If the ball is given a slight displacement with a magnet it will oscillate. An elementary theory[1] predicts simple harmonic motion with a natural period τ_0 related to γ according to

$$\gamma = 64m V_0/(d^4 P_0 \tau_0^2). \tag{1}$$

The remaining symbols in Eq. (1) can be identified in Fig. 1. When one performs an experiment to test this elementary theory, he finds that the experimental values of γ differ from the accepted values by as much as 15 percent depending upon the nature of the gas and upon the size and shape of the container. Consequently, many have attempted to improve or modify the method. For example, Rinkel[2] measured the initial drop of the ball and claimed better results than those obtained from a measurement of the period of oscillation. Brodersen,[3] using a photographic apparatus to record the displacement, obtained good results for air. Clark and Katz[4] used two gas containers connected by a cylindrical tube with a matching piston. They drove the piston electrically and collected data for a resonance curve. Their values of γ are probably the best

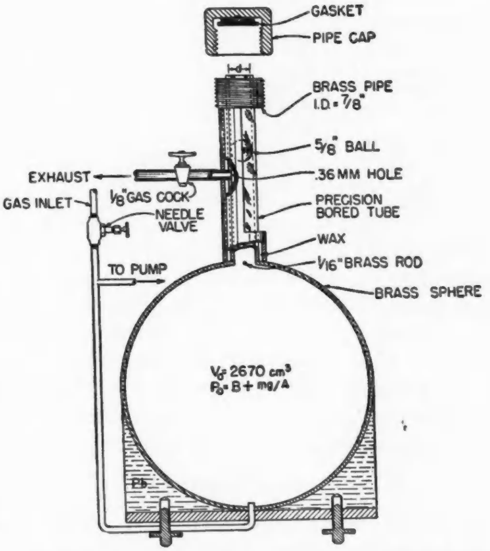

FIG. 1. Apparatus for self-sustained oscillations.

experimental values published to date. However, their apparatus is too elaborate for routine laboratory use. The disagreement between the elementary theory and the experimental results

[1] M. W. Zemansky, *Heat and Thermodynamics* (McGraw-Hill Book Company, Inc., New York, 1943), pp. 107–110.
[2] R. Rinkel, Physik. Z. 30, 805 (1929).
[3] P. H. Brodersen, Z. Physik 62, 180 (1930).
[4] A. L. Clark and L. Katz, Can. J. Research 21A, 1 (1943).

obtained with the original Rüchardt apparatus is to be expected because the observed oscillation is highly damped which precludes a precise measurement of the period. Furthermore, the measured period of this damped oscillation is not the natural period τ_0 in Eq. (1). Still further, the elementary theory neglects the effect due to heat conduction, the fact that actual gases do not obey the ideal gas law and the fact that the effective mass of the oscillator is not the mass of the ball.

The apparatus herein described is a Rüchardt apparatus with an additional feature to maintain the oscillations. The additional feature is a small hole which is drilled through the wall of the tube as shown in Fig. 1. The ball is caused to oscillate in the tube by adjusting the needle valve so that the ball rises slowly in the tube until it passes the hole at which time a small amount of gas escapes to start the oscillation.

TABLE I. Results of determining γ by self-sustained oscillations.

Gas	γ (Experimental)	γ (Accepted)	Percent diff.
CO_2	1.284 ± 0.001	1.300	1.1
NH_3	1.305 ± 0.002	1.315	1.0
O_2	1.382 ± 0.001	1.396	1.0
N_2	1.386 ± 0.001	1.403	1.2
H_2	1.345 ± 0.002	1.405	4.2
A	1.631 ± 0.001	1.670	2.3
He	1.563 ± 0.002	1.667	6.2

The escaping gas causes a drop in pressure which contributes to a periodic driving force which maintains the oscillations. The ball can be made to oscillate symmetrically with respect to the hole by finer adjustment of the needle valve. The oscillations are maintained as long as the gas passes slowly through the apparatus.

A detailed analysis[5] of this oscillator yields

$$\gamma = 64m V_0/(d^4 P_0 \tau^2) DEGH,$$

where τ is the period of oscillation, $DEGH$ is a product of four correction factors and the other symbols can be identified in Fig. 1. The quantity D corrects for the damping; E corrects for the fact that the mass of the ball is not the effective mass of the oscillator but the mass of the ball increased by a fraction of the mass of the oscillating gas; G corrects for the fact that actual gases do not obey the ideal gas law; and H corrects for the effect of the finite thermal conductivity of the gas which prevents the oscillations from taking place adiabatically. The effects due to the hole in the tube, the volume of the tube and the slow passage of the gas through the apparatus can be ascertained from the theoretical expressions for the correction factors. Briefly, the escaping gas during the upper half of the displacement cycle causes a drop in pressure and the incoming gas causes a rise in pressure. A Fourier analysis of the fluctuation in pressure about the equilibrium value shows that only the first harmonic is necessary to describe the fluctuation in pressure due to the slow passage of gas through the apparatus and that this harmonic slightly leads the displacement of the ball which provides the driving force. The magnitude of this harmonic and its phase relative to the displacement depends upon the thermal conductivity of the gas, area of the hole, size and mass of the ball, ratio of surface area to the volume of the container and the period of oscillation. The apparatus shown in Fig. 1 is designed so that each of the correction factors is approximately unity for the common gases. The design is determined from a consideration of the constants of the apparatus which appear in the theoretical expressions for the correction factors. As a general criterion, the container should have a small surface area to volume ratio, the ball should be large and dense and the hole should be small.

If one assumes that the four correction factors differ from unity by a negligible amount, the equation

$$\gamma = 64m V_0/(d^4 P_0 \tau^2) = C/(P_0 \tau^2) \qquad (2)$$

can be derived simply from fundamental principles as for the original Rüchardt apparatus. The constant C for this particular apparatus is 13.00 in. of Hg-sec^2. The results of Table I were obtained by substituting measured values of the equilibrium pressure and period of oscillation in Eq. (2). The accepted values of γ in Table I, except those for ammonia and oxygen, are those values reported by Clark and Katz.[4] These values are chosen for comparison because Clark and Katz measure γ at various values of pressure and compare their extrapolated value at zero pressure with that obtained from spectroscopic data.

[5] W. F. Koehler, J. Chem. Phys. 18, 465 (1950).

The accepted values of γ for ammonia and oxygen are taken from Euchen's[6] "Table of Most Probable Values."

The advantages of this method are as follows: (1) The oscillations can be maintained indefinitely and the period can be measured with a precision of 0.1 percent. (2) Various gases can be introduced simply into the apparatus. (3) It provides an excellent exercise for the interpretation of experimental errors. Note that all quantities in Eq. (2) can be conveniently measured to within 0.1 percent and that the final result may differ from the accepted value by several percent. Note also, that if a relative method is used, better agreement with the accepted values can be obtained.

The disadvantages of this method are as follows: (1) The apparatus is somewhat more complicated to construct than the original Rüchardt apparatus. (2) Compared to the results from high frequency methods, this method yields poorer results for gases which have a large thermal conductivity. (Note the results for hydrogen and helium.) This, however, can be said about most laboratory methods using low frequency oscillations. It is due to the fact that the thermal conductivity correction increases with the period of oscillation. (3) This method yields poorer results for gases which deviate considerably from an ideal gas than for gases which approach an ideal gas. However, the G correction factor $-(P/v)(\partial v/\partial P)$ for most of the common gases at average laboratory conditions differs from unity by less than one percent.

The following comments may be helpful during the construction and operation of the apparatus. The precision-bored glass tubes with matching ball bearings (0.001 in. diameter difference) were obtained from the Corning Glass Works for approximately $6.00 each in lots of three. The hole in the wall of the tube was drilled with a wide-angled drill having a carboloy tip. Kerosene was used as a cutting oil during the drilling process which required less than an hour, after a little practice on an ordinary glass tube. If the hole is too large the amplitude of oscillation is large and the motion is not uniform. This can be corrected without introducing appreciable error by closing the end of the exhaust with thin sheet metal in which a smaller hole has been punched with a needle. After a thorough initial cleaning the tube can be kept clean by occasionally pushing a lint-free swab moistened with acetone through the tube. The tube is placed in a vertical position with the aid of a round level placed on the upper end of the tube. When the tube and ball are clean, the tube is in a vertical position and the needle valve is adjusted so that the ball rises slowly in the tube; then, one can observe the ball spinning rapidly. The needle valve used is a Hoke Valve, No. 315. The container used is a brass sphere which was originally made for the top of a flag pole. Somewhat more satisfactory results can be obtained by using a larger container than that shown in Fig. 1 but the additional inconvenience of a larger apparatus does not warrant it.

[6] A. Euchen, *Handbuch der Experimentalphysik* **8**, 433 (1929).

Symposium on Molecular Structure and Spectroscopy

A Symposium on Molecular Structure and Spectroscopy will be held at the Department of Physics, The Ohio State University, from June 11 through June 15, 1951. There will be discussions of the interpretation of molecular spectroscopic data as well as methods of obtaining such data. In addition, there will be sessions devoted to those phases of spectroscopy of current interest. A dormitory will be available for those who wish to reside on the campus during the meeting. For further information or for a copy of the program when it becomes available, write to Professor Harald H. Nielsen, Department of Physics, The Ohio State University, Columbus 10, Ohio.

The Symposium will be sponsored this year jointly by the Graduate School and the Department of Physics and Astronomy at The Ohio State University and by the Division of Chemical Physics of the American Physical Society.

Some Experiments in Viscous Fluid Flow

KARLEM RIESS AND JOHN E. BAUDEAN
Tulane University, New Orleans, Louisiana
(Received August 14, 1950)

The motion of a viscous fluid contained between two coaxial cylinders is described. A speed criterion, although derived assuming the cylinders to have infinite length, is confirmed for the case of cylinders of finite length.

It is found that in the apparatus used there are three modes of viscous fluid motion: (a) All moving particles move in circles concentric with the axis. (b) The fluid within the annulus between the two cylinders divides itself into pairs of vortex rings in which the particles of the fluid rotate simultaneously both about the axis of the cylinders and the core of the vortex. (c) Turbulent motion appears.

The optical property of visibility of planes of equal shear in the liquid under steady flow is used to study glycerine-water and castor oil-ethyl alcohol solutions, as well as several pure oils. An explanation of the visibility of these lines is proposed. An adaptation to the measurement of viscosities is suggested.

THE problems of fluid flow stability have been studied by many workers. G. I. Taylor predicted the initial speed at which fluid motion between two concentric rotating cylinders became unstable. Assuming the disturbance to be periodic he found that the wavelength was twice the width of the annulus at the onset of instability. He also plotted streamlines of the disturbed motion and showed that they were symmetrical ring vortices of approximately square cross section. These were observed in water to which eosin or fluorescein was added.[1]

Recently W. W. Hagerty has shown that when certain concentrations of glycerin-water solutions are in a state of steady flow the planes of

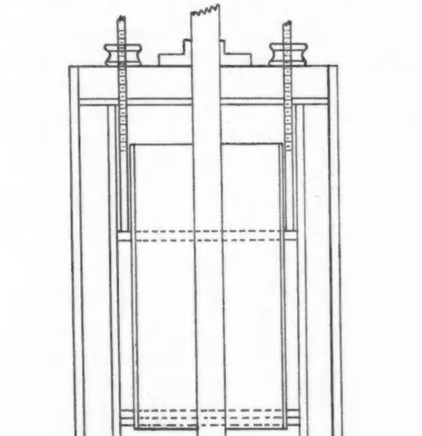

FIG. 1. Sketch of experimental apparatus: one-half full scale.

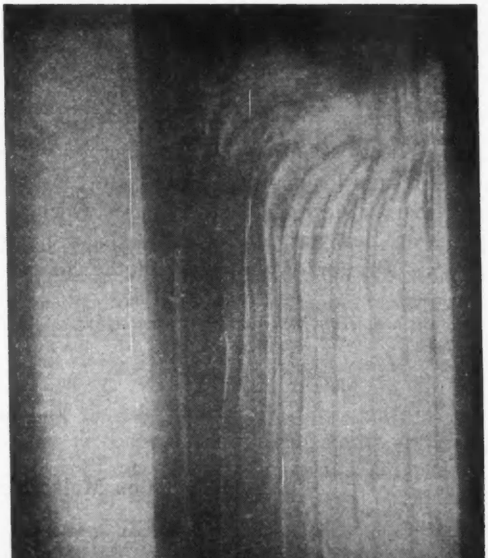

FIG. 2. Beginning of Couette motion: glycerin-water solution.

[1] G. I. Taylor, Trans. Roy. Soc. (London) **A223**, 269–343 (1923).

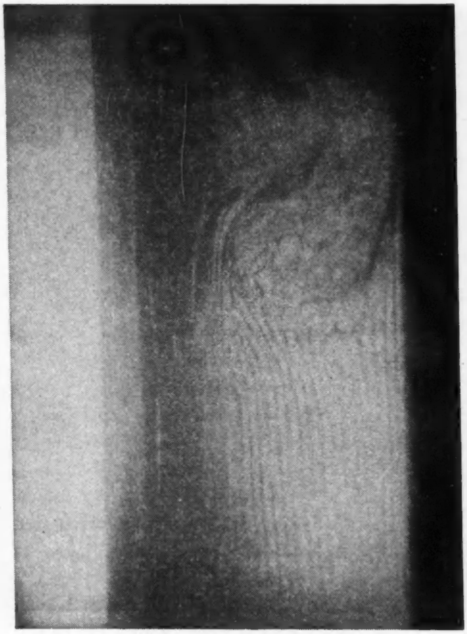

FIG. 3. Couette motion with end effect: glycerin-water solution.

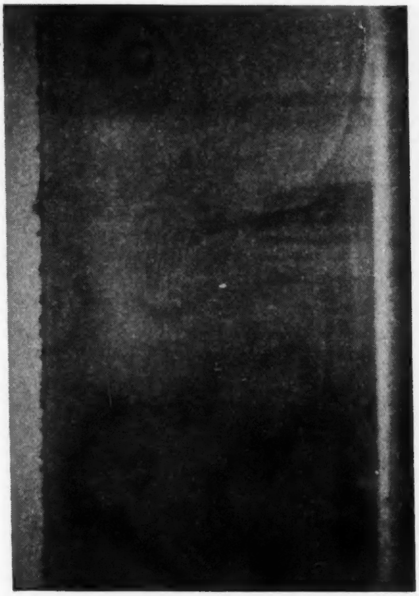

FIG. 4. Beginning of vortex motion: glycerin-water solution.

equal shear in the liquid became visible in ordinary light, without the use of dyes, if viewed along a path tangent to the shear planes.[2]

The experiments reported here were designed to correlate and extend those of Taylor and Hagerty. The validity of Taylor's criterion for annuli of finite length was studied. The effects of fixed end planes on the size, formation, and stability of the vortex systems formed in the annulus were also investigated.

EXPERIMENTAL DETAILS

The apparatus consisted of two coaxial cylinders, the inner of brass and the outer of Lucite, mounted so that the inner one was free to rotate about its vertical axis when the outer cylinder was fixed. The annulus was bounded by fiberboard planes, the upper one movable. Both cylinders were mounted in a Lucite box, which served as a temperature bath and allowed full visibility of the fluid motion. The width of the annulus was 1.340 cm and the maximum usable length 20 cm (Fig. 1).

[2] W. W. Hagerty, J. Appl. Mech. 17, 54–58 (1950).

The rotation of the inner cylinder was effected by a dc motor and pulley system. The speed of rotation was measured by means of a mechanical counter and electric timer. Viscosity measurements were made with a Sayboldt viscosimeter.

The planes of equal shear in the viscous mixtures studied were viewed in diffused indirect light against a white vellum background. The observer and camera were normal to the screen. Ordinary daylight, when not too bright or direct, was used with good results. Backlighting the vellum is also satisfactory. The camera and observing microscope were mounted separately from the cylinders. This arrangement required little adjustment when changing from visual observation to photography. Measurements were made on glycerin-water and castor oil-ethyl alcohol mixtures, as well as pure glycerin, castor oil, olive oil, and various grades of lubricating oil.

The observations began by rotating the inner cylinder at a speed below that at which the vortex motion would develop. This gave the Couette motion, a laminal flow in which each particle of the liquid rotates in a circle concentric with the cylinders (Figs. 2 and 3). As the speed of the

FIG. 5. Vortex formation: glycerin-water solution.

FIG. 6. Late stage in vortex development: glycerin-water solution.

inner cylinder is slowly increased, visual evidence of secondary motion appeared, beginning at the surface of the inner cylinder and spreading out radially across the annulus, turning down and around to form a vortex (Figs. 4 and 5). Vortices closer to the bounding plane developed more rapidly than those below. At greater distances from the bounding plane all vortices appeared to develop at the same rate.

A later stage in the development is shown in Fig. 6. The top vortex has developed from the disturbance and by its motion has caused the adjacent fluid to start the vortex motion, rotating in the opposite direction. These square vortices always exist in pairs, since no momentum is supplied or removed axially (Figs. 7 and 8).

If the speed of rotation was increased uniformly the vortex motion became unstable, and turbulent motion resulted. Taylor's relation for the critical rotational velocity for vortex motion may be written

$$\omega^2 = \frac{\nu^2\pi^4(R_1+R_2)}{2(R_2-R_1)^3R_1^2[0.0571\{1-0.652(R_2/R_1-1)\}+0.00056\{1-0.652(R_2/R_1-1)\}^{-1}]},$$

where ω is the angular velocity, ν the kinematic viscosity, R_2 the radius of the outer cylinder, and R_1 the radius of the inner cylinder. A comparison of calculated and observed values of critical velocity for glycerin-water solutions is shown in Table I. The variation of critical velocity with kinematic viscosity is shown in Figs. 9 and 10. The turbulent motion was obtained only when liquids with a low coefficient of kinematic viscosity were tested. For liquids having a high coefficient of kinematic viscosity the speeds necessary for the production of turbulent motion were above the range of the apparatus. The results obtained in the experiments verified the applicability of Taylor's criterion, which was developed initially for an annulus of infinite length.

The variation of viscosity with temperature

for glycerin-water and castor oil-ethyl alcohol mixtures was measured for a range of concentrations. Given the temperature and concentration the kinematic viscosity was obtained from the curves; or values of critical velocity for certain known temperatures were observed and the viscosity of the mixture then measured for these temperatures. For this procedure the concentration of the solution need not be known.

Table II gives the lengths of the vortices for the glycerin-water solutions. These results verify the Taylor statement that the wavelength of the secondary motion is approximately equal to twice the width of the annulus. Similar readings were made for castor oil-ethyl alcohol solutions.

If the length of the annulus is increased while the rotational speed is held constant the vortices become elongated to account for this increase in length. The reverse is true when the length is decreased. The vortices throughout the length of the annulus are of equal size, with the exception of those immediately adjacent to the end planes, which are slightly longer. If the length of the annulus was increased by an amount approximately equal to the length of one vortex, an unstable condition resulted. A readjustment took place until an even number of vortices was formed.

A polaroid analyzer was mounted in front of the apparatus and the lines viewed. No improvement in the clarity of the lines was noted. No change was observed when the incident light was polarized.

The lines became fainter when the motion of the inner cylinder was continued at constant speed for a period of ten to thirty minutes. The lines were visible for longer periods at low speeds than at high speeds. No satisfactory explanation of this effect is known. Tests indicated that the viscosity had decreased after the solution had been tested in the annulus for some time. The decrease varied from one to five percent, depending on the time of continuous testing. This small decrease in viscosity could not account for the complete vanishing of the lines.

If the system was brought to rest by suddenly

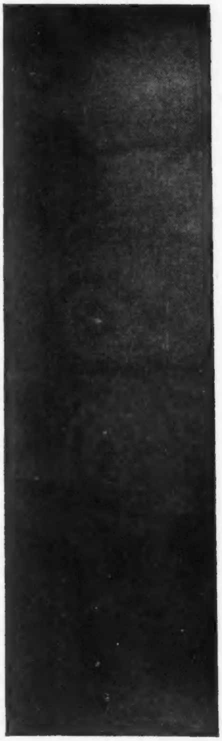

FIG. 7. Square vortices: glycerin-water solution.

FIG. 8. Square vortices: castor oil-ethyl alcohol solution.

TABLE I. Comparison of observed and calculated values of critical velocity for glycerin-water solutions.

Viscosity (stokes)	Calculated velocity (rps)	Observed velocity (rps)	Difference
0.1061	0.2126	0.224	0.0114
0.1445	0.2296	0.250	0.0204
0.1307	0.2622	0.258	0.0042
0.1331	0.2670	0.300	0.0330
0.1430	0.2868	0.333	0.0462
0.1524	0.3055	0.292	0.0135
0.1650	0.3143	0.367	0.0527
0.1653	0.3312	0.350	0.0188
0.1942	0.3895	0.375	0.0145
0.1962	0.3933	0.375	0.0183
0.2215	0.4441	0.483	0.0389
0.2326	0.4665	0.475	0.0085
0.2772	0.5558	0.600	0.0442
0.2804	0.5623	0.583	0.0207
0.3052	0.6120	0.667	0.0550
0.3711	0.7443	0.700	0.0443
0.4602	0.9228	1.000	0.0772
0.6420	1.120	1.133	0.0124
0.6687	1.340	1.358	0.0172
0.7581	1.520	1.500	0.0201
0.9993	2.003	2.025	0.0213
1.058	2.106	2.192	0.0859
1.327	2.662	2.683	0.0205
1.419	2.845	2.892	0.0472
1.533	3.074	3.116	0.0415
2.120	4.251	4.300	0.0384
2.249	4.511	4.583	0.0720
2.939	5.893	5.900	0.0064
3.477	6.972	7.016	0.0432
3.674	7.368	7.400	0.0320
4.182	8.385	8.400	0.0147

TABLE II. Dimensions of vortices for glycerin-water solutions.

Length of annulus (cm)	Number of vortices	Length of vortices (cm)
11.900	8	1.487
11.700	8	1.462
11.462	8	1.432
11.226	8	1.402
10.782	8	1.347
10.417	8	1.302
9.928	8	1.241
9.705	6	1.617
9.313	6	1.552
9.115	6	1.519
8.787	6	1.464
8.372	6	1.395
8.340	6	1.390
7.851	6	1.308
7.564	6	1.260
7.389	6	1.231
6.892	4	1.723
6.675	4	1.668
5.914	4	1.478

stopping the rotation, the lines remained visible, but became irregular and gradually sank. The effect was probably due to an actual separation of the water from the glycerin. Other mixtures tested, such as castor oil-ethyl alcohol, exhibit the same phenomenon, while pure oils do not. Pure glycerin cannot be used satisfactorily because of its hygroscopic property. When a fresh sample of glycerin was introduced into the annulus which had been previously sealed from the atmosphere no lines were visible for some time. This indicated that the lines were probably characteristic of mixtures and not of pure substances. Several pure, nonhygroscopic oils—castor oil, olive oil, and various grades of lubricating oil—were tested with negative results. In almost every case as long as the fluids in the

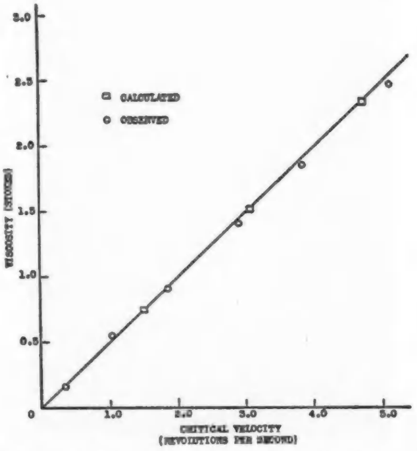

FIG. 9. Variation of critical velocity with kinematic viscosity for castor oil-alcohol solution.

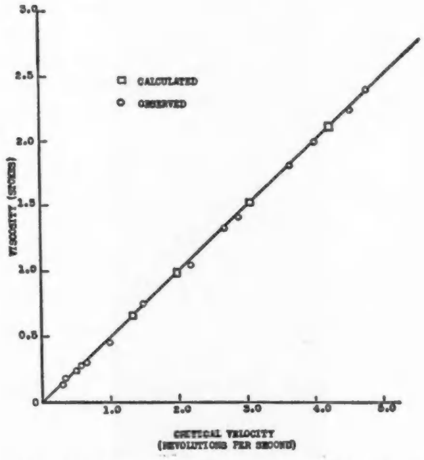

FIG. 10. Variation of critical velocity with kinematic viscosity for glycerin-water solution.

solution were imperfectly mixed the lines were visible. Although these tests are not conclusive the separation phenomenon seems to offer the best explanation of the visibility of the lines in the viscous mixtures.

To summarize, when a glycerin-water or castor oil-ethyl alcohol solution was in a state of steady shear or slowly changing rate of shear, lines appeared in the liquid which showed the fluid configuration. The number of lines observed on each area was proportional to the rate of shear in that area. The lines which appeared look something like the lines which result from an imperfect mixture of two liquids having different indices of refraction. The lines appeared in ordinary daylight, without additives in the fluid, and were not influenced by polarized light. The number of lines increased as the velocity gradient increased, and became fainter with time if the fluid was kept in a uniform condition.

Our observations have indicated that this method may be adapted to the measurement of viscosities. In certain circumstances it might offer advantages over other methods. The construction of the apparatus is simple. It may be calibrated by measuring the critical speed for a liquid of known viscosity. As with other types of viscosimeters it would be necessary to choose the dimensions of the cylinders compatible with the range of viscosities to be measured. From calibration curves and calculations the kinematic viscosity could be determined.

New Members of the Association

The following persons have been made members or junior members (*J*) of the American Association of Physics Teachers since the publication of the preceding list [Am. J. Phys. 19, 47 (1951)].

Adkins, Rutherford Hamlet, Virginia State College, Petersburg, Va.
Barnett, Judith Ann, St. Mary's Junior College, St. Mary's City, Md.
Boyd, H. Shannon, St. Ambrose College, Davenport, Iowa
Brackbill, Maurice Thaddeus, Eastern Mennonite College, Harrisonburg, Va.
Bush, Powell D. Jr. (*J*), P.O. Box 715, Emory University, Ga.
Chasteen, Joseph Wiley (*J*), 3717-E 12 St., Kansas City, Mo.
Emerson, Mary Elizabeth, 602 Prospect St., Trinidad, Col.
Fairbank, Henry A., 18 Turnor Ave., Hamden (14), Conn.
Glaser, Herman, Dept. of Physics, Texas Technological College, Lubbock, Texas
Graham, Ross, Sellersburg, Ind.
Hall, Mrs. LaVon Doner, 1539 Jefferson St., Madison, Wis.
Horn, Leon, 425 Spruce Ave., Altoona, Pa.
Hunter, Joseph Lawrence, 3901 E. Antisdale Road, South Euclid, Ohio
Huston, James Edward (*J*), 49 N. Tacoma, Indianapolis, Ind.
Kusch, Polykarp, 375 Riverside Drive, New York, N. Y.
Landis, Harry Moore, Dept. of Physics, East Central State College, Ada, Okla.
Mason, Harry, 609 N.E. 5th St., Jamestown, N. D.
Moore, Mildred Downs (*J*), Dickinson House, Mount Holyoke College, South Hadley, Mass.
Newman, John Brownie, Talladega College, Talladega, Ala.
Neville, William Thomas, Massachusetts Maritime Academy, Buzzards Bay, Mass.
Pruette, Churchill Ray, Louisburg College, Louisburg, N. C.
Rosen, James Harold (*J*), 1956 Eastern Parkway, Schenectady, N. Y.
Rozsa, John T., 12705 W. Hirst Ave., Cleveland, Ohio
Russell, Herbert O., 14632 South Holt Ave., Santa Ana, Calif.
Spear, Andrew Franklin, Jr., Lees Junior College, Jackson, Ky.
Wender, Benjamin H., 109–25 Francis Lewis Blvd., Queens Village, Long Island, N. Y.
Wiley, Roy Lee, 105 South 5th Ave., Cleveland, Miss.
Worden, David Gilbert, 22 B Pammel Ct., Ames, Iowa.

NOTES AND DISCUSSION

Proposal of the International Commission of Optics for International Standardization of Sign Conventions and Symbols in Geometrical Optics*

STANLEY S. BALLARD
Chairman, U.S.A. National Committee of the International Commission of Optics; Tufts College, Medford, Massachusetts

THE International Commission of Optics is an affiliate of the International Union of Pure and Applied Physics. Its activities are participated in by the several adhering countries through their national committees; the U.S.A. National Committee was appointed under the authority of the National Research Council. The most recent congress of the International Commission of Optics was held in London in July, 1950, and has been reported upon elsewhere.[1] At this meeting a report was received from the Czechoslovakian National Committee (Document *SO* 50–24) which listed proposals for the standardization of notations and sign conventions in optics. It was decided that this is an item of sufficient international interest and importance to demand that action upon it should be initiated. Therefore, a special working committee was appointed to draw up a set of recommendations for submission to the various national committees for comment and perhaps acceptance. This committee served under the chairmanship of J. Cojan of France, and included the following: S. S. Ballard of the U.S.A., P. Jimenez-Landi of Spain, H. König of Switzerland, L. C. Martin of Great Britain, G. Toraldo di Francia of Italy, and A. C. S. van Heel of Holland. (No delegates from Czechoslovakia were present at the congress, so that country could not be represented on this *ad hoc* committee.) At meetings held during the congress, this committee decided that the rather broad Czech recommendations should be reduced to a few specific items which would be likely to receive ready acceptance—these are listed below. In due time questionnaires will be circulated to the national committees in order to collect information on general practice in the several countries regarding the specific symbols employed in geometrical optics, and the answers to these questionnaires may serve as the basis for further recommendations.

It will be noted that the recommended sign conventions and symbols listed below are not inconsistent with American Standard *Z*10.6–1948 "Letter Symbols for Physics," which was drawn up with the assistance of the Committee on Letter Symbols and Abbreviations of the American Association of Physics Teachers. The new recommendations are currently under consideration by American Standards Association Sectional Committee *Z*58 "Standardization in Optics," of which Professor Francis W. Sears is chairman. Surely all will agree that any step in the direction of international cooperation in standardization is in itself worthwhile.

The writer of this note would welcome receiving comments, either favorable or adverse, concerning the following recommended items. All comments received will be studied by the U.S.A. National Committee before a report on this subject is made to the International Commission of Optics.

Sign Conventions

1. The axis of the optical system shall be the x-axis of the reference co-ordinates, except in certain cases where it is taken as the z-axis.
2. If there is no special reason to make another choice, the sense of the incident light shall be from left to right.
3. The radii of curvature of surfaces shall be measured from the poles; i.e., a surface convex towards the incident light will have a positive radius of curvature, and conversely.
4. The focal length on the object side shall be measured from the principal plane in the object space, and the focal length on the image side from the principal plane in the image space.
5. The distance from a point on the axis shall be taken as positive above the axis and as negative below the axis.

Symbols

6. Gothic letters shall not be used.
7. When corresponding elements in object and image space are designated by the same letter, the elements in the image space shall be distinguished by the sign prime (′).
8. Reflection shall be considered as a special case of refraction, by putting $n' = -n$.
9. Points shall be designated by capital italic letters.
10. Lengths or segments shall be designated by small italic letters.
11. Angles shall be designated by small Greek letters.

* Presented at the New York meeting of the American Association of Physics Teachers, of February 1–3, 1951.
[1] J. Opt. Soc. Am. 40, 697–8 (1950).

Mathematical Emphasis in Undergraduate Physics

JOHN PHELPS
Colorado College, Colorado Springs, Colorado

IT is generally agreed that modern physical research has shown a steadily increasing dependence upon mathematical methods. This trend has been reflected in the graduate schools as a comparison of the curricula of today with those of twenty-five years ago will show. After some discussion with Professors Paul E. Boucher and Howard Olson at Colorado College, it was decided that a survey of the extent to which the undergraduate physics program has been modified to keep pace with the new demands placed upon it might be of value in the planning of under-

graduate physics courses. There exists today a wide variation in the emphasis which is placed on mathematics for the potential graduate student in physics.

A questionnaire was designed and a copy sent to the chairman of the physics department of each of the 392 schools listed by the U. S. Office of Education as awarding at least one degree in physics between July 1, 1948, and July 1, 1949. The questionnaires were mailed out in March, 1950 and over 66 percent were returned within two months. This response is considered satisfactory. The writer takes this opportunity to express his appreciation to all of those who responded to the questionnaire.

A question naturally arises concerning the reliability of a sample of 260 responses in evaluating the collective opinion of all physics curriculum planners. In the absence of evidence to the contrary we may assume that the results are representative of the opinions of physics department heads in general. The reliability of a sample of 260 may be computed by statistical formulas which show that, for example on question 8, the probability is 0.9 that the sample results do not differ by more than 5 percent from the actual collective opinion.

The questionnaire is reproduced below in compressed form, together with composite results.

1. Institution: state-supported schools, 60; liberal arts colleges, 119; endowed universities, 66; technical schools, 15.
2. Approximate total number of undergraduate majors now in your department: 6268 (total from 260 responses).
3. Approximate number of these planning to do graduate work in physics: 2546 (total from 244 responses).
4. Does your department offer a course in mathematical physics or its equivalent for undergraduates? yes, 26.6 percent; no, 73.4 percent (259 responses).
5. If such a course is offered, what texts are used? (Frequently listed were Houston, Page, Slater and Frank, and Joos.)
6. Does your department offer a course in vector analysis for undergraduates? yes, 31.6 percent; no, 68.4 percent (256 responses).
7. What pure mathematics courses beyond ordinary calculus are required of your majors? Please check: differential equations, 74 percent; advanced calculus, 45 percent; theory of equations, 11 percent; complex variable, 5 percent; hyperbolic functions, 1 percent; advanced algebra, 9 percent. (Many listed some type of survey course in mathematics touching on topics of value to physics students.)
8. Do you believe that mathematics is receiving sufficient emphasis as a part of the undergraduate physics curriculum at most schools? yes, 29.6 percent; no, 70.4 percent (189 responses).
9. Do you favor the teaching of mathematical physics as a separate undergraduate course? yes, 62.7 percent; no, 37.3 percent (228 responses).

The figures from questions 2 and 3 together indicate that 40.6 percent of present undergraduate majors plan to do graduate work in physics. It was hoped that the approximate percentage of majors in a given department planning to do graduate work might reveal something about the instructional objectives of that department. In interpreting the answers to question 4 "mathematical physics" was restricted to mean a course designed to give the student some understanding of the fundamental mathematical methods of physics. Generally, "theoretical physics" courses were included in this category while survey courses in mathematics were not. Two out of three schools answering "yes" to question 6 indicated that the vector analysis course was taught in the mathematics department. Many replies listed one or more of the courses under question 7 as "optional" or "recommended." The figures from question 8 are interesting and, it is hoped, of some significance. They seem to indicate a somewhat widespread dissatisfaction with existing physics curricula in this particular respect.

A space was provided at the end of the p inted questionnaire for comments, several of which may be of interest. Each of those listed here appeared, in one form or another, on at least a dozen responses. Many were quite emphatic in their opinion that mathematics was not receiving enough emphasis in undergraduate physics, and many expressed themselves strongly in favor of a mathematical physics course. Of those who opposed the introduction of more mathematics or a mathematical physics course there were several who felt that the approach in undergraduate physics should be experimental and descriptive, and several who thought that a mathematical physics course would be a step towards too much undergraduate specialization. Some smaller schools stated that their main reason for not offering such a course was lack of staff. In a number of cases the poor mathematical background of graduating seniors in physics was traced to a deficiency in high school mathematics and the necessity of postponing the elementary calculus course to the sophomore year. Many expressed a real concern over the "elementary" character of much of undergraduate physics. A number felt that they were handicapped by insufficient cooperation between physics and mathematics departments.

Sensitive Arrangements of the Wheatstone Bridge

CHARLES WILLIAMSON
Carnegie Institute of Technology, Pittsburgh, Pennsylvania

IN years gone by, when galvanometers of low sensitivity and batteries of low emf were the only ones commonly available, it was important to know what arrangement of the Wheatstone bridge was the most sensitive. Nowadays, with modern equipment, any hastily assembled arrangement is usually good enough; but sometimes it is not. Theory or experiment must then be invoked for guidance.

Two well-known treatments of bridge sensitivity are given in books by Starling[1] and Page and Adams.[2] In these references it is shown that sensitivity may be increased by making the bridge currents as large as prudence will permit, by approximately equalizing all four of the bridge arms, and by choosing a galvanometer of resistance comparable to that of the bridge itself; it is also shown that not much will be gained by further refinements of detail. In such discussions it is often assumed that a galvanometer coil can be designed to fit the bridge, and that sensitivity is limited by that bridge arm which can be varied by the smallest fraction of itself. These assumptions may be helpful to the theorist, but hardly to the experimental man; he must choose between only two or three galvanometers, but he may shunt any bridge arm with a variable high resistance, thus obtaining as small a change as he pleases. The following questions are then pertinent:

1. With a given galvanometer, what is the most sensitive arrangement of the equal-arm bridge?
2. When a bridge contains two high resistance arms and two low resistance arms, what is the most sensitive arrangement: (a) with the galvanometer connected from the junction of the high-resistance arms to the junction of the low resistance arms? (b) with the galvanometer connected otherwise?
3. How do these optimum arrangements compare in sensitivity?
4. If it is necessary to depart from any optimum arrangement in a specified manner, how much sensitivity is lost?

The answers to all these questions can, of course, be obtained by mathematical methods; but the task is so tedious that it seems worth while to do the work experimentally. Accordingly, the results presented in this paper were obtained in the laboratory.

Four 4-dial resistance boxes were used for the bridge arms. The "galvanometer" was a 1.5-milliampere Weston Model 45 meter having a resistance of 300 ohms. The bridge emf was taken from a 150-ohm dual slidewire rheostat connected across the 110-volt dc power line. The maximum power dissipation in any arm was made exactly 0.2 watt by suitably adjusting the bridge emf. Then each arm in turn was unbalanced by an amount dR such that the meter indicated a current of 0.1 milliampere; the quotients dR/R were found and the four quotients averaged. Then a sensitivity S, defined by the equation $S=(dR/R)_{\min}/(dR/R)$, was calculated.

In the accompanying tables the results are generalized by listing the bridge arms under columns headed A and B as multiples of the galvanometer resistance R_g. Table I shows that the best equal-arm bridge is that with arm resistance equal to that of the galvanometer; the maximum is flat, half-sensitivity occurring for arm resistance less than one-tenth or more than ten times galvanometer resistance. This conclusion applies when it is possible to choose a bridge emf that will give maximum safe dissipation in any bridge arm. Sometimes circumstances require the use of a bridge emf so small that it will not overload

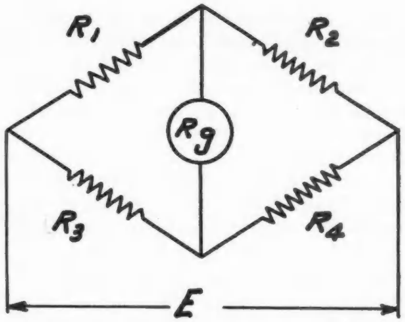

FIG. 1(a). Wheatstone bridge circuit. E, bridge emf; R_g, resistance of galvanometer; R_1, R_2, R_3, R_4, resistances of bridge arms.

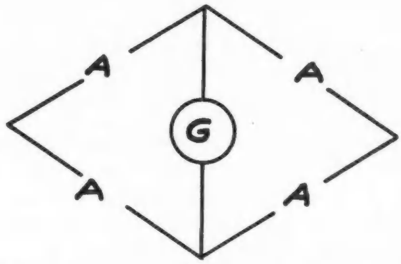

FIG. 1(b). Arrangement pertaining to Table I. The symbol A denoting generalized arm resistance is defined by the equations $R_1=R_2=R_3=R_4=R$, and $A=R/R_g$. The generalized galvanometer resistance G is unity.

TABLE I. Relative sensitivities of specified arrangements of the equal-arm bridge. See Fig. 1(b) for circuit diagram. E, bridge emf; R, resistance of bridge arm; R_g, resistance of galvanometer (here it is 300 ohms); A, generalized resistance of bridge arm, defined by the equation $A=R/R_g$; G, generalized resistance of galvanometer (here it is unity); dR/R, fractional unbalance in bridge arm producing a galvanometer current of 0.1 ma; S, sensitivity of circuit, defined by the equation $S=(dR/R)_{\min}/(dR/R)$; see text for details.

R (ohms)	A	E_1 (volts)	dR/R	S_1	E_2 (volts)	dR/R	S_2
30	0.1	4.9	0.0264	0.63	4.9	0.0264	0.63
60	0.2	6.9	0.0234	0.71	4.9	0.0333	0.50
150	0.5	11.0	0.0183	0.91	4.9	0.0405	0.41
300	1.0	15.5	0.0167	1.00	4.9	0.0520	0.32
600	2.0	21.9	0.0174	0.96	4.9	0.0793	0.21
1500	5.0	34.6	0.0223	0.75	4.9	0.151	0.11
3000	10.0	49.0	0.0277	0.60	4.9	0.277	0.06

any reasonable bridge arm. When that is the situation, Table I shows that the lower the arm resistance is, the better.

Tables II and III deal with bridges composed of equal pairs of resistances. Maxwell's rule prescribes that the galvanometer is to be connected from the junction of the higher pair to that of the lower pair, provided that the galvanometer has a higher resistance than the battery. Table II presents that case. It is seen that the optimum resistance for the variable pair B depends on that of the fixed pair A; if the bridge emf can be made optimum, the best arrangement is with all four arms equal, whether they match the galvanometer or not; if the bridge emf must be

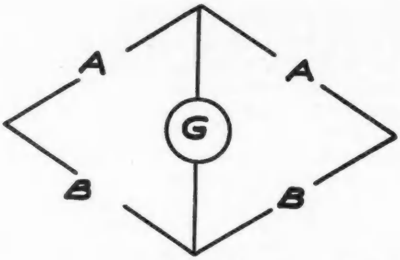

FIG. 2. Arrangement pertaining to Table II. Here $A = R_1/R_g = R_2/R_g$ and $B = R_3/R_g = R_4/R_g$.

TABLE II. Relative sensitivities when bridge arms are paired in accordance with Maxwell's rule. See Fig. 2 for circuit diagram. Symbols used in diagram and in column headings have the same meanings as heretofore, except as follows: $A = R_1/R_g = R_2/R_g$; $B = R_3/R_g = R_4/R_g$.

	A = 0.1		A = 1.0				A = 10.0			
B	E_1	S_1	E_1	S_1	E_2	S_2	E_1	S_1	E_2	S_2
0.1	4.9	0.63	4.9	0.40	4.9	0.40	4.9	0.11	4.9	0.11
0.2	4.9	0.58	6.9	0.56	4.9	0.40	6.9	0.15	4.9	0.11
0.5	4.9	0.[illegible]0	11.0	0.83	4.9	0.37	11.0	0.24	4.9	0.11
1.0	4.9	0.39	15.5	1.00	4.9	0.32	15.5	0.32	4.9	0.10
2.0	4.9	0.28	15.5	0.83	4.9	0.26	21.9	0.44	4.9	0.10
5.0	4.9	0.18	15.5	0.50	4.9	0.16	34.6	0.58	4.9	0.08
10.0	4.9	0.10	15.5	0.33	4.9	0.10	49.0	0.60	4.9	0.06

small, it is best to have one pair of resistances low, whether the other pair is low or high.

Table III shows what happens when Maxwell's rule cannot be complied with. Again, of course, the equal-arm arrangement is the best, if the bridge emf can be made optimum; for then Maxwell's rule is not violated. If the bridge emf must be small, the equal-arm arrangement is

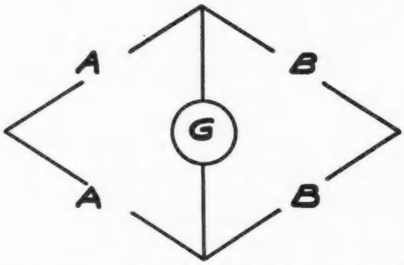

FIG. 3. Arrangement pertaining to Table III. Here $A = R_1/R_g = R_3/R_g$ and $B = R_2/R_g = R_4/R_g$.

TABLE III. Relative sensitivities when bridge arms are not paired in accordance with Maxwell's rule. See Fig. 3 for circuit diagram Symbols used in diagram and in column headings have the same meanings as heretofore, except as follows: $A = R_1/R_g = R_3/R_g$; $B = R_2/R_g = R_4/R_g$.

	A = 0.1				A = 1.0				A = 10.0			
B	E_1	S_1	E_2	S_2	E_1	S_1	E_2	S_2	E_1	S_1	E_2	S_2
0.1	4.9	0.63	4.9	0.63	8.5	0.30	8.5	0.30	24.7	0.11	24.7	0.11
0.2	5.2	0.50	4.9	0.47	9.3	0.50	8.5	0.46	25.0	0.18	24.7	0.18
0.5	6.5	0.42	4.9	0.32	11.6	0.72	8.5	0.53	25.7	0.32	24.7	0.31
1.0	8.5	0.29	4.9	0.17	15.5	1.00	8.5	0.55	27.0	0.42	24.7	0.38
2.0	11.4	0.22	4.9	0.09	16.4	0.83	8.5	0.43	29.5	0.53	24.7	0.44
5.0	17.6	0.15	4.9	0.04	20.7	0.59	8.5	0.24	36.5	0.56	24.7	0.38
10.0	24.7	0.10	4.9	0.02	26.9	0.43	8.5	0.14	49.0	0.60	24.7	0.30

still the best when the arm resistance of the fixed pair A is less than or equal to that of the galvanometer; but when the fixed pair is higher, the variable pair B should have a resistance lying between that of the fixed pair and that of the galvanometer.

There is one more question that should be asked; to answer it, new data are needed. In many laboratories there are several galvanometers that are identical except for coil resistance. From the data presented in the foregoing tables it is evident that an unknown resistance should first be measured approximately, and then determined precisely with a bridge and a galvanometer both of which are near to it in resistance. If the proper galvanometer is not chosen, how much sensitivity is sacrificed? This question is answered in Table IV. The data were obtained as follows.

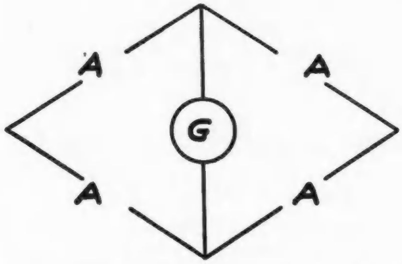

FIG. 4. Arrangement pertaining to Table IV. Here $G = R_g/R$ and A is unity.

TABLE IV. Relative sensitivities of the equal-arm bridge when the arms are not changed but several different galvanometers are used. These galvanometers are identical except for coil resistance. See Fig. 4 for circuit diagram. Symbols used in diagram and in column headings have the same meanings as heretofore, except as follows: G, generalized resistance of galvanometer, defined by the equation $G = R_g/R$; A, generalized resistance of bridge arm (here it is unity).

G	E	S	$S(G)^{\frac{1}{2}}$
0.01	15.5	2.0	0.2
0.1	15.5	1.9	0.6
1.0	15.5	1.0	1.0
10	15.5	0.19	0.6
100	15.5	0.02	0.2

An equal-arm bridge was set up, the arm resistance being equal to that of the galvanometer. Then the resistance of the galvanometer was changed by connecting suitable resistors in series or in parallel with it. The relative sensitivities of different arrangements are recorded under S in Table IV, due regard being paid to the changes in current sensitivity of the galvanometer itself when shunted. Now it is easily shown[1,2] that the current sensitivities of galvanometers that are identical except for coil resistance are proportional to the square root of the coil resistance. Therefore the effective sensitivity for the purpose of this investigation is not S but $S(G)^{\frac{1}{2}}$; accordingly, this is the quantity recorded in the last column of Table IV. There is no heading S_2 in Table IV because the bridge emf could not be increased progressively; for the bridge arms remained constant.

It is seen that little sensitivity is lost by a moderate

mismatch between the galvanometer and the unknown in an equal-arm bridge. Even when the mismatch is as bad as 0.01 or 100, the sensitivity is 0.2 that of the optimum arrangement.

It should be noted that caution must be used in applying the maximum safe emf to any bridge. Preliminary balancing should be done with low emf, and the maximum should be carefully calculated and applied for a short time only, if possible.

The results presented in these tables may be useful in designing bridges containing vacuum-tube impedances or wire-wound strain gauges; they may also apply to certain ac bridges, particularly those used at audiofrequencies.

[1] Starling, *Electricity and Magnetism* (Longmans, Green and Company, London, 1929), p. 71.
[2] Page and Adams, *Principles of Electricity* (D. Van Nostrand Company, Inc., New York, 1931), p. 175.

Yo-Yo Technics in Teaching Kinematics

IRVING L. KOFSKY
Syracuse University, Syracuse, New York

A TOY popular with children and with which many students are familiar is the Yo-Yo.[1] Very few textbooks make mention of it, and I have nowhere seen even a qualitative discussion of how such a top works. However, several important physical principles can be illustrated, at least qualitatively, by a demonstration with this toy. Sufficient skill in operation of the Yo-Yo can be gained by about an hour's practice, and there will usually be at least one person in the class who can do more advanced "tricks," some of which may illustrate principles not shown by the simpler exercises. Perhaps at the expense of being thought eccentric, I have mastered enough tricks to present to my sophomore physics students an informal demonstration which they seemed to think quite interesting.

Although almost everybody has seen a Yo-Yo top, I shall describe it here. The Yo-Yo itself is a balanced double-wheel-and-axle-type body suspended by a string looped around the axle. Specifically, the one I use is made from a wooden cylinder 5.5 cm in diameter and 3 cm in height. The edges of the cylinder are rounded and a 2.5-mm slit is cut in the center perpendicular to the axis, to such a depth as to leave an axle 6 mm in diameter. The mass of this particular Yo-Yo is 53 g. A fine string 230 cm long is doubled upon itself and twisted, and the axle rides in the loop that is formed at the end. (Other Yo-Yo's are made of plastic or metal, and some, not as well suited for demonstration purposes, simply have the string fastened to the axle. My model, made by Duncan, retails for 35 cents.)

The other end of the string is fastened, usually by a double overhand knot, to either the middle or index finger, and the string is then wound up on the axle. When the Yo-Yo is thrown or even dropped downward, it will unwind itself down the string and continue to rotate about its axis, or "sleep" (the loop of string slipping on the axle), until a sharp pull on the string causes the top to roll itself and the string back into the operator's hand. Needless to say, a reasonably thorough discussion of this down-and-up movement would bring in nearly all of the elementary concepts of translational and rotational motion. Two qualitative questions that should make the student think are: When the string is jerked while the Yo-Yo is sleeping, why does the top catch and roll up? and, What is the effect of using a Yo-Yo of larger moment of inertia, on the time that it will remain sleeping, before friction effects cause the spinning to cease? Certainly the teacher should experience no difficulty in framing quantitative questions about the motion.

A careful observation of the sleeping Yo-Yo reveals that its motion entails two further subjects of study in elementary classes. For one, the top acts as an extremely simple gyrostat: the string does not untwist when the body has angular momentum about its principal geometrical axis, but after the rotation is allowed to die out the hanging Yo-Yo slowly starts to rotate about a vertical axis under the influence of the torsional torque provided by the unwinding string. Secondly, the impulses generated by imperfections in the axle cause the sleeping top to set up standing waves in the vertical string: as the frequency of the driver decreases—that is, as the Yo-Yo slows down because of friction—a different mode of vibration is excited for each (wideband) resonance between top and string.

A trick called "Around the World" is a subject for discussion of other rotational effects. The Yo-Yo is thrown out forward from the waist, along the horizontal, and is made to revolve in a vertical circle with center at the operator's hand. The top then has both orbital and rotational angular momentum. If it is thrown with sufficient force the cylinder will still have enough rotational kinetic energy to roll itself up the string after the circle is described. Perhaps here one might be lucky enough to have the string break (it has a tendency to wear out at the axle) showing that the Yo-Yo flies off *tangential* to its original path! "Walking the Dog" is another easily-learned trick of physical interest. The sleeping top is allowed to just touch the ground, and its rotational motion is partly changed to translational as the edge of the cylinder rolls, with slipping, along the floor. In a variation of "Walking the Dog," the Yo-Yo is swung far forward and touched to the ground, while the string, which is kept taut, is lowered until it is horizontal. The top will bounce gently while spinning at the end of the string, its angular velocity in such a direction that, were the string released from the finger, it would roll away from the hand. A pull on the string causes the Yo-Yo to bounce toward the operator jerkily, as the string rolls up on the axle. Here two opposing torques are at work: the tension in the string at its lever arm and the intermittent reaction force of the ground at the edge of the "backspinning" top. This latter torque is applied as a series of angular impulses of short duration, one each time the Yo-Yo touches the ground.

When the top is sleeping at the bottom of the string, it has an amazingly high angular velocity. Consider a Yo-Yo of mass m and radius r. The moment of inertia I of this nearly cylindrical body is roughly $\frac{1}{2}mr^2$. If all the rotational kinetic energy of the sleeping Yo-Yo could be changed into the work necessary to raise the weight mg through a height h, then

$$\tfrac{1}{2}I\omega^2 = mgh, \tag{1}$$

where ω is the initial angular velocity. Therefore,

$$\omega = 2(gh)^{\frac{1}{2}}/r. \tag{2}$$

A Yo-Yo that just rolls up a 1-meter string must have had an angular velocity of at least 36 rev/sec. Of course, the energy necessary to raise the string and to overcome the friction losses would increase this figure.

As a check, the angular velocity of the sleeping Yo-Yo was measured with a stroboscope. When the toy was given a large initial kinetic energy by being thrown down sharply, angular velocities as high as 110 rev/sec were measured; since the stroboscope could not be adjusted before the Yo-Yo had slowed down somewhat because of friction, it is not inconceivable that 140 rev/sec is often reached.

The high energy of such a rapidly rotating top can be utilized in a trick guaranteed to cause a mild sensation in the classroom. Throw the Yo-Yo down as sharply as feasible and, while it sleeps, release the string from the finger to which it is attached, and grasp the top of the string between the thumb and index finger of the left hand, palm down. A sharp slap on the back of the left hand and release of the string as soon as the Yo-Yo has caught causes the top to fly up the string, and, now spinning more slowly, to reach a level several feet above the head, with the string all wound up. This presents a rather difficult dynamic situation to which, of course, Eq. (1) is not strictly applicable. Catching the Yo-Yo in a coat pocket on the way down is a spectacular way to wind up the demonstration.

[1] Trademark registered U. S. Patent Office 300504 by Donald F. Duncan, Inc., Chicago, Illinois. Other manufacturers produce essentially the same product. There is evidence that the toy was known to the Athenians as *Disc* and to 17th and 18th century England and France as *Bandalore* and *L'Emigrette*. Yo-Yo is a popular sport in the Philippines, and the Duncan people, by far the largest manufacturers, used to advertise their product by importing Filipino experts to give demonstrations in five-and-ten-cent stores and hold contests among children in neighborhood theaters.

A Simple Camera for Taking X-Ray Powder Patterns at Elevated or Reduced Temperatures

L. F. Connell, Jr., and H. C. Martin, Jr.
University of Texas, Austin, Texas

THE important role of x-rays in the study of the fundamental properties of matter is widely recognized and most colleges and universities have one or more x-ray sources. Use is made of these sources in laboratories in atomic or molecular physics as well as in research and in courses devoted entirely to the study of crystal structure. The study of samples by x-ray diffraction techniques is usually limited to samples at room temperature, although many types of furnaces have been devised for elevating the temperature of the sample and a few for reducing it. These furnaces and the associated film holder are usually quite expensive to construct or buy and require extensive calibration if the temperatures are to be known with reasonable accuracy. This is true because the large temperature gradients within the camera, combined with the small volume of the customary x-ray powder sample, make temperature measurement with any of the direct-measuring devices inaccurate (as with a thermocouple) or impractical (as with a resistance or expansion thermometer).

A simple, easily constructed furnace which provides accurate temperature determination has been devised to use a liquid bath to maintain sample temperature. The furnace allows recording of the back-reflection diffraction lines (Bragg angle > 75°) and has been found extremely useful for accurate determination of lattice constants over a wide temperature range.

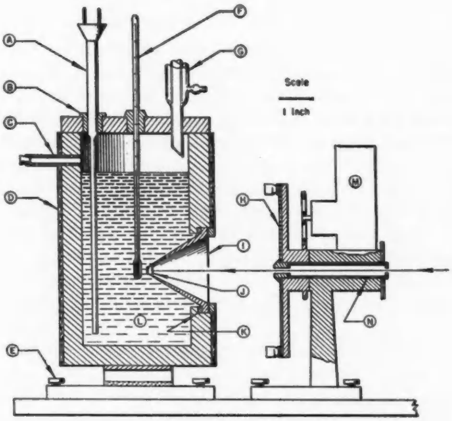

Fig. 1. Cross-sectional view of camera. A, 500-w blade heater; B, lead stopper; C, overflow outlet; D, asbestos paper; E, locking screw; F, thermometer; G, water-cooled reflux condenser; H, x-ray film; I, removable heat shield; J, powder sample; K, lead gasket; L, liquid; M, motor to rotate film holder; N, collimator tube.

The tapered-wall aluminum cone of the camera (Fig. 1) is a few mils thick near the apex, where the powder sample is packed. The aluminum foil heat shield I is one mil thick and has a small entrance hole for the direct x-ray beam. Additional heat shielding can be provided by means of a second foil of aluminum (or cellophane for low temperatures) placed outside and parallel to the first. Although the furnace is most easily used at fixed temperatures which are boiling or freezing points of various substances, it may be used at other temperatures by regulating the heating rate. Use of the camera at low temperatures requires precautions to prevent moisture condensation inside the sample holder.

The film holder and collimator utilized here in conjunc-

tion with this furnace are part of the back-reflection component of a General Electric X.R.D. unit; however, any one of a number of types of back-reflection arrangements may be used. With the G. E. film holder and sector mask, three separate exposures of opposing 60° sectors can be made on a five-inch circular film, providing a diffraction record at three different temperatures and thus reducing film shrinkage error in comparing the patterns for different temperatures.

The results obtained during more than a year of using this system have shown it to be more nearly accurate in temperature determination and control than furnaces suitable for use in cylindrical powder cameras, and to be more suitable for use by relatively inexperienced personnel.

A Modified Cotton Balance*

ZABOJ V. HARVALIK
Institute of Science and Technology, University of Arkansas†
Fayetteville, Arkansas

TEACHERS of general physics laboratory courses often have felt that an experiment demonstrating the interaction of a magnetostatic field with a conductor through which an electric current flows would be desirable. The author constructed a simple apparatus demonstrating quantitatively the interaction of a permanent magnetic field and a magnetic field produced by an electric current using Cotton's[1] approach. The apparatus consists of a permanent magnet of large-area pole shoes and narrow air gap, and a conductor movably located in the air gap. It uses a damping magnet of a watt-hour meter as permanent magnet and a wire as conductor bent into a rectangular loop.

Assuming that a direct current I passes through the wire loop, a part of which is located inside of the magnetic field of the permanent magnet and having a length of l, a force will develop of the intensity

$$F = IlB, \tag{1}$$

where l is the length in centimeter of the wire inside of the magnetic field, B the flux density of the magnetic field in maxwell, and I the current intensity in abampere. According to the direction of the current and considering that the conductor wire is horizontally mounted, the wire will be pushed upward or downward when the current is permitted to flow. If one connects the conductor to a battery in such a way that the force acting upon the wire causes the wire to move upward against the gravitational field of the earth, the lifting force can be compensated by attaching weights to the wire. These weights represent a force

$$F = gm, \tag{2}$$

where g is the acceleration due to gravity (980 cm sec^{-2}) and m the mass of the attached weights. Therefore, one can establish the equation

$$gm = IlB. \tag{3}$$

If I is known, the equation can be solved for

$$B = gm/Il. \tag{4}$$

Description of the Instrument

A wooden block WB is mounted on a 6×12-inch base plate BB (Fig. 1), and carries two binding posts BP to

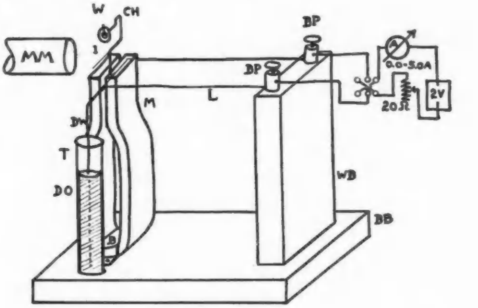

FIG. 1. Modified cotton balance.

which the wire loop L is attached. The wire loop L is of rectangular shape, the shorter side of which is placed into the air gap of the permanent magnet M. At the center of the smaller side of the wire loop, a carrying hook CH is soldered as well as an indicator wire. The carrying hook CH is of the same gauge as the wire loop (24 gauge, 0.51-mm diameter). The indicator wire is of 38 gauge (0.1-mm diameter), and serves in connection with a measuring microscope as compensation indicator. The weights added to the carrying hook WC are fiber washers W whose masses must be determined with an analytical balance. The permanent magnet M is mounted on the board by a brace B of appropriate shape.

Owing to the fact that undesired vibrations of the wire loop L occur, a damping device is introduced. The damping device consists of a damping wire DW soldered onto the loop L. The damping wire dips into a test tube T filled with oil of viscosity 10 as damping liquid DO.

Experimental Procedure

After the student wires the circuit according to the diagram (Fig. 1), and establishes the zero position of the loop by observation through the measuring microscope MM, he tests the upward direction of the force on the loop by switching on the current. A weight washer W is then placed upon CH and the electric current is regulated with the resistor (20 ohm, 5 amp) till the zero position of the indicator I is established, using the measuring microscope. The current read on the ammeter is recorded. A 2nd, 3rd, 4th, etc., weight washer W is added and the procedure is repeated. A graph indicating the proportional relationship between the current and the compensating weights is prepared (Fig. 2) and the flux density within the air gap of the magnet is calculated by formula 4.

Students will have no difficulty in performing the measurements which are accurate to one percent if an ammeter

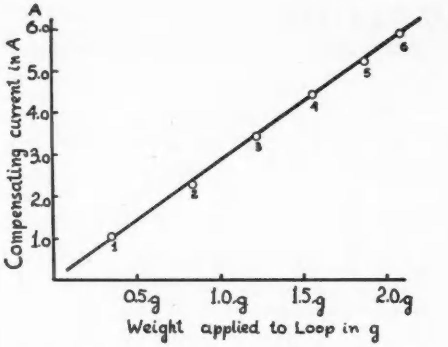

FIG. 2. Relationship between current and compensating weights.

with 0.1 amp reading is used. A simplified construction of somewhat different design of the instrument described above was used in a general physics laboratory course at the University of Prague, Department of Physics, Prague, Czechoslovakia before World War II.

* This instrument was exhibited at the Colloquium of College Physicists, State University of Iowa, Iowa City, Iowa, June, 1947.
† Research Paper No. 934, Journal Series, University of Arkansas.
[1] A. A. Cotton, J. Phys. 9, 384 (1900).

A Basic Principle for the Telescope and Microscope

H. C. SCHEPLER
University of California, Berkeley, California

AND

A. N. SMITH
Emerson Electric Manufacturing Company, St. Louis, Missouri

WHEN introducing telescopes and microscopes in our physics and optics courses, the characteristic features of each instrument are usually described. Thereafter, they are too often thought of as unrelated pieces of equipment except that light passes through each of them. Although the individuality of each instrument must be emphasized, the close relationship between them will be imparted to the student if he is first introduced to a basic principle from which both instruments can be derived. The usual prerequisites, including the formation of real and virtual images, magnification, and an understanding of accommodation in the eye should forego this presentation.

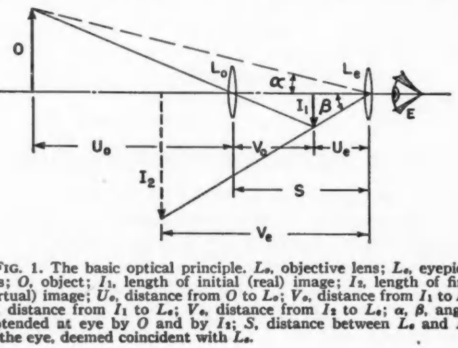

FIG. 1. The basic optical principle. L_o, objective lens; L_e, eyepiece lens; O, object; I_1, length of initial (real) image; I_2, length of final (virtual) image; U_o, distance from O to L_o; V_o, distance from I_1 to L_o; U_e, distance from I_1 to L_e; V_e, distance from I_2 to L_e; α, β, angles subtended at eye by O and by I_2; S, distance between L_o and L_e; E, the eye, deemed coincident with L_e.

The basic optical principle concerned can be demonstrated from Fig. 1. Lens L_o forms an image I_1 of an object O. In either a telescope or a microscope this image is viewed with the eye through an eyepiece L_e. Angular magnification is used in describing the performance of visual instruments. A general expression for magnification is derived as follows.

When considering the linear image I_2 and the linear object O, in Fig. 1,

$$M=\tan\beta/\tan\alpha, \tag{1}$$

where M is the angular magnification. Now

$$\tan\beta=I_2/V_e, \quad \text{and} \quad \tan\alpha=O/(S+U_o). \tag{2}$$

By similar triangles

$$I_2=(V_e/U_e)I_1, \quad \text{and} \quad I_1=(V_o/U_o)O. \tag{3}$$

From Eqs. (1)–(3)

$$M=\frac{I_2/V_e}{O/(S+U_o)}=\frac{S+U_o}{U_e}\frac{V_o}{U_o}. \tag{4}$$

This equation defines the magnification in terms of linear axial distances that apply to the general two-lens system shown.

In a telescope the angular subtense of the final image is compared with that of the object viewed at a distance. The object O is at ∞; S is negligible compared with U_o, and Eq. (4) becomes

$$M=V_o/U_e. \tag{5}$$

Since $U_o=\infty$, $V_o=f_o$, and $U_e=f_e$, where f_o is the focal length of the objective and f_e is the focal length of the eyepiece. Then Eq. (5) becomes

$$M=f_o/f_e. \tag{6}$$

This is the familiar expression for the magnification of a telescope.

In a microscope the angular subtense of the final image is compared with that of the object viewed by the naked eye at the distance of distinct vision, D. (This distance is generally taken as 10 in. or 25 cm.) In this case

$$D=S+U_o. \tag{7}$$

Substituting Eq. (7) into Eq. (4) we find that

$$M=DV_o/U_eU_o. \tag{8}$$

For the usual condition in a microscope, image I_2 is located at distance D from the eye, so $D=V_e$. Then $D/U_e=V_e/U_e=m_e$, where m_e is the magnification due to the eyepiece. Similarly, $V_o/U_o=I_1/O=m_o$, where m_o is the magnification due to the objective. Then Eq. (8) becomes $M=m_em_o$. This is the familiar expression for the magnification of a microscope. The relationship between the telescope and microscope has thus been established by deriving the expression for the magnification of each from a common general equation.

LETTERS TO THE EDITOR

A Teaching Device

MOST physics teachers use multiple-response quizzes and examinations in the first physics course. Over the years I have built up a goodly stock of these covering the usual gamut of general-physics subject matter. Personally, I quiz with great frequency, since quizzes, in my judgment, are excellent teaching devices *if they are gone over in the class*. And I use multiple-choice questions in abundance, since they permit a much closer coverage of the material than either problem quizzes or the exposition type. Now after ten or twelve weeks in the course, when the students have been exposed to some six or eight or more of these multiple-choice quizzes, I begin a series of assignments which, in my own experience, pay substantially in the learning of the student.

I announce the following:—"Gentlemen, you are now fairly familiar with multiple-choice quizzes. You have acquired, we hope, a reasonable capacity for critical thinking. For every assignment henceforth you will bring to class five multiple-choice questions which you construct yourself. Minimize problems and definitions. Concentrate on theory. State the proposition clearly and find five 'good' answers, *one of which is correct*. Go anywhere in the material we have covered and where your own interest dictates."

This note is intended to report on the exceedingly great profit which accrues to the student by this device. As a matter of fact, the students themselves pointedly admit that it is one of the best *learning* devices to which they are exposed. Obviously, it takes some sound thinking to formulate the question and some very exacting analysis to find "good" answers. By "good" we do not necessarily mean correct, since, obviously, four of the answers are indeed wrong. In the beginning, the performance of the students does not look very promising. Good multiple-choice questions are hard to concoct. But after a few weeks of it, I encounter numerous quiz questions which I can profitably use on later classes. Indeed, this is what I do!

I have kept close watch on the class performance throughout several years of this endeavor. It appears safe to report that a student's achievement on a quiz is substantially enhanced by this device. He is immeasurably better equipped to handle an examination on those sections he has worked over in this way.

Dillard University
New Orleans, Louisiana
JULIUS SUMNER MILLER

Demonstration of Emission and Absorption of Sodium Vapor

THE following experiment shows in a very simple and easy way absorption and emission of sodium vapor.

The light of a sodium lamp passes a stop, a condenser C and a second lens L, so that on a screen S a yellow circular spot is visible in dark surroundings. Between C and L a Bunsen burner is put, so that the lens L gives a sharp image of the funnel of the Bunsen burner upon the screen S, as in Fig. 1.

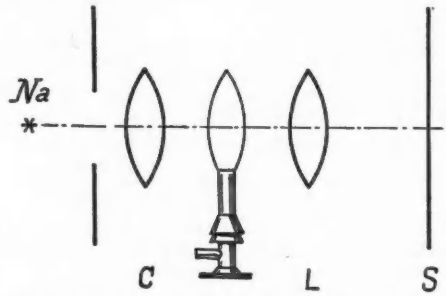

FIG. 1. General disposition of source, Bunsen burner, and screen.

The light of the common blue-colored flame of the burner is too weak to be visible on the screen (Fig. 2a). Coloring the flame with sodium causes that part of the flame appearing within the yellow spot on the screen to become dark (absorption) while the part beyond the spot becomes visible as a yellow emission image (Fig. 2b). If potassium, for example, is used instead of sodium the flame can be examined in a similar way, but on the screen the part within the spot is now also colored (emission only, no absorption) as in Fig. 2c.

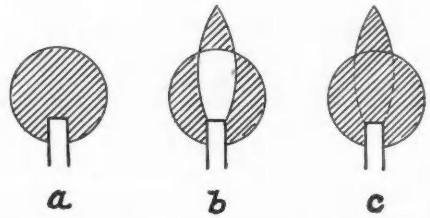

FIG. 2. Appearance of screen with (a) ordinary Bunsen flame, (b) flame colored with sodium, and (c) flame colored with potassium.

Institute of Physics
University of Vienna
Austria
F. BLAHA

Testing the Rayleigh Resolving Power Criterion

AN exceedingly direct technique for comparing the distinguishability of two light sources imaged by a telescope with the Rayleigh resolving power criterion (two sources will be resolved if the central diffraction maximum of one is at or beyond the first diffraction minimum of the other) has been used for several semesters in the student optics laboratory.

A thin calcite crystal backed by a sheet of metal foil in which a *single* pinhole (or slit) has been punched produces *two* apparent point (or slit) objects. The two effective sources have the same size and shape. Light transmitted by them has the same intensity, but is plane-polarized at right angles so that no interference between them is directly observable. In each of these respects, this method of producing the double object is superior to two distinct pinholes in a sheet of foil.

In practice, a telescope which can be rotated delicately in a horizontal plane is equipped with an iris diaphragm at the objective lens. The foil is illuminated by green light filtered from a high pressure mercury arc. The calcite is rotated until the line joining the holes is horizontal. The iris diaphragm is closed until the "dumbbell" nature of the object can just be detected. A polarizer (oriented to extinguish light from one pinhole) is inserted. The cross-hair intersection is set on the first minimum of the observed diffraction pattern. The iris diaphragm is opened and the polarizer is removed.

Usually at this time, the observer finds the cross-hair intersection near the other pinhole image but beyond it. The utility, the convenience, and the arbitrariness of the Rayleigh resolution criterion are thus established.

Johns Hopkins University
Baltimore 18, Maryland
FRANK MOONEY

The Teaching of Physics to Premedical Students

As it is now nearly thirty years since Queen's University inaugurated a program of two years' instruction in physics for medical students, fellow physicists may be interested in a brief report on the nature and the success of the plan. At the outset it should be stated that at Queen's University there are no premedical years as such, instruction in the fundamental sciences being given in the early years of a six-year medical course. Teaching in physics, therefore, is given to groups of "full-fledged" medicals, who constitute units quite independent of classes in the Faculties of Arts and of Applied Science. It is realized that in most universities physics is taught to prospective medicals when in their premedical years, but the program described below is equally applicable to this more common arrangement.

In the first year, all work in electricity and magnetism is removed from a course which otherwise covers topics usually discussed in general physics. The course consists of three lectures per week throughout the year with associated laboratory work of two hours per week. The omission of electricity and magnetism from the general course has a twofold advantage. It provides time to cover adequately *and quantitatively* the other main branches of physics, and it gives more time for a discussion of applications of special interest to medical students. Anyone who has made a specialty of teaching such students knows what a wealth of applications are available in mechanics, properties of matter, heat, sound, and light. In mechanics and properties of matter, for example, one thinks of pressure measurement and the sphygmomanometer, motion of fluids and blood-flow, elasticity and pulse, surface tension and motion of the amoeba, diffusion and osmosis, and many other topics. Again, in heat, there are such obvious illustrations of fundamental principles as heating and ventilation, cooling systems and evaporation, humidity, and control of body temperature. But regarding such applications there is no need to labor the point by dealing with other branches of physics.

In the first half of the second year, instruction is given in elementary electricity and magnetism, adequate time being available, even in a course of two lectures per week, for a proper grounding in this important branch of physics, as well as for emphasizing applications in medicine. With this preparation the student has the background necessary for the second half of the second-year course. This consists of a thorough, although elementary, treatment of such topics as x-ray apparatus, x-ray tubes, properties of x-rays, high frequency currents, radioactivity, artificial radioactivity and isotopes, and the elements of nuclear physics. The student is led naturally from his work in electricity and magnetism to the discussion of these topics, whose importance to him is obvious; and he acquires a greater appreciation of and respect for fundamental physics. To give a single example, the subject of induced currents leads at once to a discussion of alternating currents and transformers for the operation of x-ray tubes, to the measurement and control of high potential differences, to the meaning of rectification, and to other related questions. Full details of the work given in the second half of the second-year course may be found in the writer's *Radiology Physics*,[1] a text based on the lectures given to that class.

Associated with the lectures is a laboratory course of two periods per week in which a group of more or less standard experiments in electricity and magnetism are combined with some of special value for this course, such as experiments dealing with the thermonic emission of electrons, radioactivity, decay of thorium emanation, the Eindhoven galvanometer, the motor, and impedance.

After a trial of over twenty-five years it may be said without hesitation that this two-year program has been an unqualified success. For many years physics has been a bugbear for medicals. Our experience has shown that many students in this group learn to like the subject, and all to respect it. In the writer's opinion, a two-year program in physics should be the normal requirement for all premedicals.

Queen's University
Kingston, Ontario
J. K. ROBERTSON

[1] J. K. Robertson, *Radiology Physics* (D. Van Nostrand Company, Inc., New York, 1941 and 1948).

Maxwell's Thermodynamic Relations

ABOUT a year or two ago, a series of letters[1–7] appeared in this Journal on the question of a mnemonic device for Maxwell's thermodynamic relations. A particularly simple mnemonic scheme has been suggested by L. A.

Turner.[6] However, this scheme is based on a notation which uses E for the internal energy, whereas many people have received their introduction to thermodynamics from books (such as Zemansky's well-known text)[8] which use U for this quantity. It is somewhat of a nuisance mentally to change E to U, and the change adds an extra element to remember in the mnemonic. Under these circumstances, it is convenient to modify Turner's scheme to

which can be read: "The good physicist has studied under very fine teachers." If one subscribes to different prejudices, it can be read: "The good physicist has studied under very few teachers." The rule of signs does not involve U (or E) and thus remains as given by Turner.

The motivation for these remarks rests on Professor Turner's comments on the catalytic action of a little foolishness.

5443 Overdale Drive
Los Angeles 43, Calif.
JOHN J. GILVARRY

[1] C. M. Focken, Am. J. Physics 16, 450 (1948).
[2] W. E. Haisley and J. Bugosh, Am. J. Physics 17, 91 (1949).
[3] H. C. Brinkman, Am. J. Physics, 17, 170 (1949).
[4] R. E. Payne, Am. J. Physics 17, 225 (1949).
[5] C. M. Focken, Am. J. Physics, 17, 225 (1949).
[6] L. A. Turner, Am. J. Physics, 17, 397 (1949).
[7] F. H. Crawford, Am. J. Physics, 17, 450 (1949).
[8] M. W. Zemansky, *Heat and Thermodynamics* (McGraw-Hill Book Company, Inc., New York, 1937).

Remarks on the Use of Fitch's Apparatus for the Measurement of the Thermal Conductivity of Thin Slabs of Poorly Conducting Materials

THE author has tested Fitch's apparatus[1] in hundreds of experiments and found it to give unreliable results, agreeing among themselves with an average deviation of about seven percent. Specimens of the same material but of different thickness give results which increase with the thickness. I have tried this with different thicknesses of rubber dental dam (got by folding), also with different thicknesses of Pyrex plate (from the same melt). Also, if one takes a 28-minute run and plots the successive 7-minutes' observations, the results are highest for the first 7 minutes and get progressively lower in the other three runs (possible because the receiver loses heat through its imperfect insulation). In no case have I got results agreeing with experiments made on the same specimens under steady conditions. I should like to hear the comments of other users of this apparatus and also of those who have used Harvalik's modification.[2]

University of Toronto
Toronto 5, Canada
JOHN SATTERLY

[1] A. L. Fitch, Am. Phys. Teacher 3, 135 (1935). Worthing and Halliday: *Heat* (John Wiley and Sons, Inc., New York, 1948), p. 182. Central Scientific Company, Chicago. Cat. J-142, p. 1169.
[2] Z. V. Harvalik, Rev. Sci. Instr. 18, 815 (1947).

AMERICAN JOURNAL OF PHYSICS VOLUME 19, NUMBER 2 FEBRUARY, 1951

ANNOUNCEMENTS AND NEWS

Book Reviews

Relativity Physics. Second edition. W. H. McCREA. Pp. 87, 10×17 cm. Methuen and Company, Ltd., London, 1947. Distributed in the United States by John Wiley and Sons, Inc., New York. Price $1.25.

This new edition of a well-known monograph in the Methuen series presents a treatment of the special theory of relativity theory in a fairly standard manner. However, apart from a discussion of the Lorentz transformation, and of kinematics, mechanics, optics, and electromagnetic theory, which of necessity must appear in such a treatise, the author treats more complex problems in a simple way. Thus the fields produced by an arbitrarily moving charge are determined, and after a brief discussion of quantum theory the Compton effect and possible disintegrations of a free particle are outlined. Finally, compressed in this little book is a discussion of thermodynamics and statistical mechanics in relativity theory.

H. C. CORBEN
Carnegie Institute of Technology

The Special Theory of Relativity. HERBERT DINGLE. Pp. 94, $4\frac{1}{4}\times 6\frac{3}{4}$ in. Methuen and Company, Ltd., London, 1940. Distributed in the United States by John Wiley and Sons, Inc., New York. Price $1.25.

The author rightly feels that it is of interest to add another book to the literature on relativity theory because the treatment offered here is new. Dr. Dingle is concerned mainly with developing relativity theory from a redefinition of the measurement of length and with examining the effect of relativity theory on the fundamental concepts of physics. The approach is a stimulating one, solidly based on classical experimental facts rather than formal algebra. Unfortunately statements like "The general expression for ds^2 represents all possible mechanical systems described in terms of all possible systems of coordinates" appear to be a little too sweeping. One would also have welcomed a statement about inertial and noninertial observers to qualify the correct but misleading "remedy for a skin disease on the organism of physical theory" that "there is no meaning in absolute acceleration."

H. C. CORBEN
Carnegie Institute of Technology

Response of Physical Systems. John Dezendorf Trimmer. Pp. ix+268, Figs. 93. John Wiley and Sons, Inc., New York, August, 1950. Price $5.00.

This is a readable introduction to the dynamics underlying the fields of servomechanisms and automatic controllers, or the subject of "Instrumentation," to use the author's expression; an introduction to the dynamics, not to the fields themselves; an introduction suitable for either the physicist intending to enter the fields, or for one who is merely trying to extend his general knowledge of various interesting branches of physics.

The author is singularly successful in writing a book which is understandable by physicists in general, not merely by those already in the field which he is trying to describe. The many definitions and concepts, and the important distinctions between related terms and concepts, are developed carefully and methodically. The limitations and implications of the customary linear approximations are explained fully. The problem of specifying the order of a dynamical system is discussed. It is shown that higher order systems may always be resolved (but not always uniquely) into component first- and second-order systems. The last chapter is on nonlinear systems.

The mathematical methods employed are those common to most fields of physics. Instead of a table of Laplace transforms, found in most books on servomechanisms, there are tables of solutions to the differential equations of interest. This certainly adds to the ease of reading and to the clarity of presentation for the general reader. As explained by the author himself: ". . . The decision (to use differential equations instead of Laplace transforms) was not due to any lack of appeal of the elegant Laplace method, but to a strong feeling that a thorough familiarity —the familiarity that is gained only by prolonged, detailed experience—with the classical method is an absolutely essential part of the foundation to be laid. It is only in the classical view of these simple linear systems that, at the present stage of the subject's development, the interplay of physics and mathematics can be followed with such instructive explicitness. To miss any of this by concentration on a method of 'working problems faster' would be a poor trade.

"Of course, the Laplace transformation is more than a method of working problems rapidly. A fuller appraisal of it . . . is given in Appendix 1. Thus, I have no quarrel with the use of the Laplace method, so long as it is presented as a supplement to the classical method, and not as a substitute for it, and so long as the classical method also receives some of the same striving or the perfection of detail which many writers have lavished on the Laplace method."

The problems and examples are drawn from many different fields of physics. However, the description and discussion of some of these problems are not as clear and as painstaking as are explanations of the underlying theory. Figure 6.14, p. 131, for example, is, as drawn, almost meaningless. Certainly, the input signal to the control amplifier should be shown explicitly.

Graphical methods for depicting the response to sinusoidal inputs are described and illustrated, but here again, these charts lack the care of preparation exhibited in other parts of the book. To say the least, the function and argument should be expressed in words, not merely by algebraic symbols. Figure 4.4 is a log-log plot of the gain of a second-order system *vs* ratio of the frequency of input to the natural frequency, but the entire accompanying discussion would apply equally well to the Cartesian plot; the significance of the slope, −12 db/octave, characterizing the high frequency ends of the curves shown is not even mentioned. Polar plots of the reciprocal of the gain *vs* the phase shift are shown in Figs. 4.6, 6.12, and 6.13, and their usefulness explained, but the transfer loci, or polar plots of the gain itself, are not mentioned. The equivalent of Nyquist's criterion for stability is developed on pp. 125–128, but the conventional form or statement of the criterion is not given. Perhaps the author was merely trying to avoid something which merely becomes a blind rule of thumb in the hands of the unthinking user.

On p. 114, the first approximation to the real root of a certain cubic equation is given as -1.97×10^3; then, the next approximation is given as -2.0×10^3. Certainly, the closer approximation should be given to the greater number of places.

On p. 120, the well-known Principle of the Equivalent Generator, actually due to Helmholz, is called "Thevenin's Theorem."

On p. 169, there is propounded the fallacy called Chauvenet's Criterion. Although the fallacy of Chauvenet's rule should be obvious after a little study of mathematical statistics, this is the second recent book, written by a physicist and published by Wiley to contain this item. For mature discussion of the problem raised on p. 169, one might read, amongst other things, a recent paper by Grubbs.[1]

In short, Professor Trimmer's book is the easily read introduction, suitable as general reading for advanced undergraduate seniors and beginning graduate students. The treatment is elegant and general, but it is not the broad, philosophical treatment claimed by the publishers. Attempts to link the treatment with cybernetics, or to apply it to biological or to sociological systems, do the book more harm than good.

Ralph Hoyt Bacon
The Perkin-Elmer Corporation

[1] Grubbs, Ann. Math. Statistics **21**, 27 (1950).

Microwave Electronics. John C. Slater. Pp. 406+xiv, Figs. 91. D. Van Nostrand Company, Inc., New York, 1950. Price $6.00.

Microwaves may be defined as electromagnetic waves with wavelengths of the same order of magnitude as the dimensions of their circuit elements. Electronics deals with the motion of free electrons under the influence of electric and magnetic fields. *Microwave Electronics* is concerned with the motion of free electrons (or other charged particles) in microwave electromagnetic fields.

The book *Microwave Electronics* is the first comprehensive and unified treatment of this field. The first half of

the book develops the theory of microwave transmission and microwave circuit elements including particularly cavities with various numbers of outputs and the periodically loaded wave guide. In the latter half, after a chapter on the "Principles of Electronic Devices," the klystron, linear accelerator, cyclotron, betatron, F-M cyclotron, synchrotron are considered in order. The treatment of each of these specific devices is well integrated with the preceding theory. Useful comparisons and contrasts are made among the various devices.

The approach throughout the book is very general. For example, much of the theory of wave guides is developed without reference to any particular shape of guide although a distinction is made between guides bounded by a single surface and by more than one surface. The completeness of the approach is also illustrated by the discussion of the response of various circuit elements to low frequency waves as well as to microwaves. Frequently the general theoretical approach is merely outlined and references to more detailed treatments are given.

It may be surprising to find such low frequency devices as cyclotrons, betatrons, and synchrotrons included in a treatise on microwaves. Their principles of operation, however, are very similar to the microwave devices insofar as they involve an interaction between charges (frequently electrons) moving (often in bunches) under the simultaneous influence of oscillating electromagnetic fields and of constant or slowly varying magnetic fields. As an example, electron bunching in a klystron is compared to that in a linear accelerator and a traveling-wave amplifier. It is clearly shown that phase stability in a linear accelerator is obtained only at the expense of defocusing of the beam. Later when F-M cyclotrons and synchrotrons are discussed, it is pointed out clearly how the magnetic field makes possible both phase and lateral stability.

Microwave Electronics is a book one reads with enthusiasm. Imbedded in each general treatment one finds conclusions of surprising generality and usefulness. The book is, however, definitely not one for the beginner in either microwaves or electronics, both because the presentation goes from the general to the specific and because it is unusually comprehensive and compact. For one with a reasonable background in electromagnetic theory and some previous knowledge of microwave devices, this treatise should serve as an integrator of this knowledge and as a very useful reference for use in advanced work in microwave electrons.

SHERWOOD K. HAYNES
Vanderbilt University

Atomic Physics. WOLFGANG FINKELNBURG. Pp. 498+x, Figs. 226. McGraw-Hill Book Company, Inc., New York, 1950. Price $6.50.

It is hardly possible nowadays to pick up a book review list without finding mention of a new book on atomic physics or a closely allied subject. The vastly increased public interest in atomic and nuclear physics in the years following the war has of course provided some of the stimulus for the expansion in this field. Another factor which greatly increases the need for such new books is the continuous and rapid progress which is being made by the research workers and the theorists. So rapid has this progress been that it is well-nigh impossible for a new book to keep abreast of the latest discoveries and theories. However, the author of *Atomic Physics* has succeeded admirably in presenting an up-to-date account of the recent developments and modern theories in this field in a well organized book.

At first glance, the reader may wonder at the choice of subject matter, because the chapter headings include not only the familiar material on "Atomic Spectra and Atomic Structure," "Atomic Theory According to Quantum Mechanics" and "Nuclear Physics," but also a chapter on "Molecular Physics" and one on "Atomic Physics of the Liquid and Solid State." However, Dr. Finkelnburg states in his preface that "Atomic physics as treated in this book, deals with our entire knowledge of the structure of matter from the elementary particles up to the solid state," and he has kept this aim consistently in view. There is a resultant de-emphasis on certain subjects, e.g., a very limited discussion on kinetic theory, and the restriction of the subject of x-ray spectra to eight pages and that of cosmic rays to thirteen pages. In addition to the five main chapter headings mentioned above, there is a very short introductory chapter which serves more or less as an extended preface. A second short chapter entitled, "Atoms, Ions, Electrons, Atomic Nuclei, Photons," which serves to introduce some of the necessary background material for the more detailed discussions which follow is somewhat sketchy and not always clear.

The book is written for students at the senior or graduate level who have had the necessary preliminary courses in elementary and intermediate physics. It is a translation from the German *Einfuhrung in die Atomphysik* and is equivalent to the forthcoming revised second German edition. An attempt is made by the author to stress meaning of experiments and theories rather than mathematical or experimental details, and in fact, mathematical derivations are limited mainly to chapters 3 and 4. It is doubtful if this omission of the mathematical details will make the text any easier for the student, since a more thorough understanding of the subject is usually gained in the process of working out mathematical problems. Instructors who like to assign problem material will find it difficult to do so because there are no problems given in the book and very few equations upon which to base problems. No references are given in the text to original reports on research work although there is an excellent bibliography at the end of each chapter containing references to books and review articles on the subject matter of that chapter.

The illustrations in the book are in general clear and to the point and will contribute much to the student's understanding. The order of presentation of the subject matter is logical and follows historical development in the main. There is very little digression from the topic under discussion, though this leads the author into difficulties over subjects which must be mentioned for completeness but cannot be discussed until later on. The reader will probably

be somewhat irked by the large number (as high as five on one page) of references to material which will be discussed later or has been discussed on another page.

One fault which this book has in common with a great many other textbooks in the field is the failure to state the units which must be used with each equation and especially with those which permit numerical calculations. This practice is particularly confusing to students when it involves equations concerning the passage of ions through electric or magnetic fields. For example, the author gives the well-known equation, $R=mcv/eH$, in which electrostatic units are used for e, yet a few lines further on the value of e/m is given in the practical system of units. In another instance, an equation is given equating ergs and cm^{-1} as used in an energy-level diagram. This will give trouble to those students who have been drilled in the rule that valid equations must have the same dimensions in all terms.

Many of the explanations given in the book are necessarily sketchy and incomplete because of the breadth of subject material which must be covered. This will make it a rather difficult text for student use and will necessitate a thorough knowledge of the background material by the instructor. However, we believe that it successfully coordinates the various domains of atomic physics and that it will be a valuable addition to the well-known International Series.

RALPH B. BOWERSOX
Jet Propulsion Laboratory
California Institute of Technology

Books Received

Antennas. JOHN D. KRAUS. 553 pp. McGraw-Hill Book Company, Inc., New York, 1950. $8.00.

Chemical Thermodynamics. FREDERICK D. ROSSINI. 514 pp. John Wiley & Sons, Inc., New York, 1950. $6.00.

Die Elektromagnetische Schirmung in der Fernmelde und Hochfrequenztechnik. HEINRICH KADEN. 274 pp. Springer-Verlag, Berlin, Germany, 1950. DM 38.

The Dome. E. BALDWIN SMITH. 228 pp. Princeton University Press, Princeton, New Jersey, 1950. $7.50.

Economic Aspects of Atomic Power. SAM H. SCHURR AND JACOB MARSCHAK. 289 pp. Princeton University Press, Princeton, New Jersey, 1950. $6.00.

Educators Guide to Free Slidefilms. (Second Edition.) MARY FOLEY HORKHEIMER AND JOHN W. DIFFOR. 128 pp. Educators Progress Service, Randolph, Wisconsin, 1950.

Electromagnetic Fields, Theory and Application. ERNST WEBER. 590 pp. John Wiley & Sons, Inc., New York, 1950. $10.00.

Electrons and Holes in Semiconductors. WILLIAM SHOCKLEY. 558 pp. D. Van Nostrand Company, Inc., New York, 1950. $9.75.

Foundations of Aerodynamics. A. M. KUETHE AND J. D. SCHETZER. 374 pp. John Wiley & Sons, Inc., New York, 1950. $5.75.

Fundamentals of Optics. (Second Edition.) FRANCIS A. JENKINS AND HARVEY E. WHITE. 647 pp. McGraw-Hill Book Company, Inc., New York, 1950. $7.00.

Fundamentals of Acoustics. LAWRENCE E. KINSLER AND AUSTIN R. FREY. 516 pp. John Wiley & Sons, Inc., New York, 1950. $6.00.

Fundamentals of Quantum Mechanics. ENRICO PERSICO (translated and edited by Georges M. Temmer). 484 pp. Prentice-Hall, Inc., New York, 1950. $6.00.

A German-English Dictionary for Chemists. (Third Edition.) AUSTIN M. PATTERSON. 541 pp. John Wiley & Sons, Inc., New York, 1950.

Handbook of Experimental Stress Analysis. M. HETENYI. 1077 pp. John Wiley & Sons, Inc., New York, 1950. $15.00.

How to Pass College Entrance Tests. ALISON S. PETERS. 192 pp. Arco Publishing Company, New York, 1950. $2.50.

The Identification of Molecular Spectra. (Second Edition, revised.) R. W. B. PEARSE AND A. G. GAYDON. 276 pp. John Wiley & Sons, Inc., New York, 1950. $8.50.

Industrial Instrumentation. DONALD P. ECKMAN. 396 pp. John Wiley & Sons, Inc., New York, 1950. $5.00.

The Inelastic Behavior of Engineering Materials and Structures. ALFRED M. FREUDENTHAL. 587 pp. John Wiley & Sons, Inc., New York, 1950. $7.50.

The Laplace Transformation. WILLIAM TYRRELL THOMSON. 230 pp. Prentice-Hall, Inc., New York, 1950. $3.75.

The Meaning of Relativity. (Third Edition.) ALBERT EINSTEIN. 162 pp. Princeton University Press, Princeton, New Jersey, 1950. $2.50.

Negative Ions. (Second Edition.) H. S. W. MASSEY. 136 pp. Cambridge University Press, New York, 1950. $2.50.

The Next Development in Man. L. L. WHYTE. 255 pp. New American Library, New York, 1950. $.35.

Our Natural Universe Including Man. PERCY A. CAMPBELL. 75 pp. College Offset Press, Philadelphia, Pennsylvania, 1950. $2.00.

The Philosophy of Edmund Montgomery. MORRIS T. KEETON, 386 pp. University Press, Dallas, 1950. $5.00.

The Philosophy of Mathematics. EDWARD A. MAZIARZ. 286 pp. Philosophical Library, New York, 1950. $4.00.

Physics, Its Laws, Ideas and Methods. ALEXANDER KOLIN. 890 pp. McGraw-Hill Book Company, Inc., New York, 1950. $6.50.

Principles and Applications of Waveguide Transmission. GEORGE C. SOUTHWORTH. 689 pp. D. Van Nostrand Company, Inc., New York, 1950. $9.50.

Procedures for Evaluating Research Personnel With a Performance Record of Critical Incidents. 42 pp. American Institute for Research, Pittsburgh, 1950.

Radio Communication at Ultra High Frequency. J. THOMSON. 203 pp. John Wiley & Sons, Inc., New York, 1950. $4.50.

Response of Physical Systems. JOHN D. TRIMMER. 268 pp. John Wiley & Sons, Inc., New York, 1950. $5.00.

Sourcebook on Atomic Energy. SAMUEL GLASSTONE. 546 pp. D. Van Nostrand Company, Inc., New York, 1950. $2.90.

The Theory and Practice of Industrial Research. DAVID BENDEL HERTZ. 385 pp. McGraw-Hill Book Company, Inc., New York, 1950. $5.50.

Theory of the Interior Ballistics of Guns. J. Corner. 443 pp. John Wiley & Sons, Inc., New York, 1950. $8.00.

Traveling Wave Tubes. J. R. Pierce. 260 pp. D. Van Nostrand Company, Inc., New York, 1950. $4.50.

Ultrasonic Coagulation of Phosphate Tailing. Dudley Thompson. 77 pp. Virginia Polytechnic Institute, Blacksburgh, Virginia, 1950. $.75.

Wellenmechanik. Karl Jellinek. 304 pp. Wepf & Co., Basel, Switzerland, 1950. 34 francs.

Weltsystem Weltather und die Relativitatstheorie. Karl Jellinek. 450 pp. Wepf & Co., Basel, Switzerland, 1950. 45 francs.

AMERICAN JOURNAL OF PHYSICS VOLUME 19, NUMBER 2 FEBRUARY, 1951

RECENT MEETINGS

The Pennsylvania Conference of College Physics Teachers

The Pennsylvania Conference of College Physics Teachers met at Haverford College, Haverford, Pennsylvania on Friday and Saturday, October 20 and 21, 1950. Approximately one hundred guests attended the sessions. Dr. Richard M. Sutton, *Haverford College*, was in charge of arrangements for the meeting; he was assisted by Professor W. R. Wright. Registration and informal visiting were provided for the guests Friday afternoon at the Franklin Institute, Philadelphia. Four tours of the Franklin Museum and Planetarium were offered. Latecomers were registered at Sharpless Hall, Haverford College, where a visit of the laboratories was provided Friday afternoon.

Dr. D. A. Keys, *Chalk River Laboratory*, Ontario, lectured Friday evening in Roberts Hall on the topic "The Work of the Canadian Atomic Energy Project." Invited papers were delivered Saturday morning by Dr. Albrecht Unsöld, Director of the Observatory, Kiel, Germany, on the subject "On the Origin of Cosmic Rays" and by Dr. W. F. G. Swann, Director of the Bartol Institute, Swarthmore, Pennsylvania, on the subject "On the Teaching of Physics." Eight contributed papers completed the program Saturday morning. The conference dinner was served to guests Friday evening in Founder's Hall.

Contributed Papers

Useful search coils and systems for uniform magnetic fields. Milan W. Garrett, *Swarthmore College.*

Dynamic hysteresis loop tracer. T. A. Benham, *Haverford College.*

Physics in the news today. Pearl I. Young, *The Pennsylvania State College, Pottsville Center.*

Demonstration of interference of light waves. Gordon M. Dunning, *Indiana State Teachers College.*

Modes of vibration of a rotating string. John S. O'Connor, S. J., *St. Joseph's College.*

The Geiger-Nuttall law and regularity in alpha-radioactivity. F. W. Van Name, *Franklin and Marshall College.*

Undergraduate experiments in modern physics. W. C. Elmore, *Swarthmore College.*

Experiments with nuclear emulsions. M. Elaine Toms, *University of Pennsylvania.*

Michigan Teachers of College Physics

The Michigan Teachers of College Physics held its fall meeting on Saturday, October 28, 1950, at the University of Michigan, Ann Arbor. Seventy-two teachers of physics from colleges and universities throughout the state registered for the meeting. The morning sessions were held in the West Physics Building. Professor E. F. Barker, *University of Michigan*, presided at the session. An invited paper, "Age determinations by Radioactivity," was presented by Professor H. R. Crane, *University of Michigan*. A paper, "Circular Translation," by Professor W. W. Sleator, *University of Michigan*, was deferred until a later meeting due to the length of the program.

Luncheon was served to the group at the Michigan Union. In the afternoon the Illumination Laboratory of the Engineering Research Institute was opened for inspection by the visitors. Dr. R. A. Boyd demonstrated the work of the Laboratory, particularly in connection with the illumination of school rooms. Randall Laboratory was also made available for inspection by visitors.

Dr. Everett R. Phelps, *Wayne University*, invited the Michigan Teachers of College Physics to his institution for its spring meeting.

Program

Invited Paper: Age Determinations by radioactivity. H. R. Crane, *University of Michigan.*

Solution of the Schroedinger equation for an approximate atomic field. E. H. Kerner, *Wayne University.*

The tinplate experiment for sophomore physics. D. A. Naymik, *Michigan State Normal.*

Nondestructive testing of manufactured parts. G. P. Brewington, *Lawrence Institute of Technology.*

Demonstration of research on natural illumination in school rooms. R. A. Boyd, *University of Michigan.*